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On Stability and Evolution of Solutions in General Relativity

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ON STABILITY AND EVOLUTION OF SOLUTIONS

IN GENERAL RELATIVITY

by

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ABSTRACT

ON STABILITY AND EVOLUTION OF SOLUTIONS

IN GENERAL RELATIVITY

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This thesis is concerned with several problems in general relativity and low energy string theory that are pertinent to the time evolution of the gravitational field. We present a formulation of the Einstein field equations in terms of variational techniques borrowed from geometric analysis. These equations yield the evolution equations for the Cauchy problems of both general relativity and low energy string theory. We then proceed to investigate the evolutionary linear stability of Schwarzschild-like solutions in higher dimensional relativity called black strings. These objects are determined to be linearly unstable. This motivates a further stability analysis of the charged p-brane solutions of low energy string theory. We show that one can eliminate linear instabilities in p-branes for sufficiently large values of charge.

We also consider the characteristic problem of general relativistic magnetohydrodynamics (GRMHD). We compute the eigenvalues and eigenvectors of

GRMHD and establish degeneracy conditions. Finally, we consider the initial value problem for axisymmetric GRMHD. We formulate the general Einstein and MHD equations under the assumption of a stationary axisymmetric spacetime without assuming the circularity condition.

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NOTATION

The following are common notations in the text:

∇_a - the covariant derivative with respect to the Levi-Civita connection

Γ^a_{cd} - Christoffel symbols with respect to the Levi-Civita connection

G - Newton's constant

G_{ab} - the Einstein tensor

iff - if and only if (logical equivalence)

M - n-dimensional C^∞ Lorentzian manifold

η_{ab} - four dimensional Minkowski metric

$\mathbb{R}^n$ - Euclidean n space

$v_a v^a$ - indicates implicit Einstein summation convention over a

R - the scalar curvature

R_{ab} - the Ricci curvature tensor

R_{abcd} - the Riemann curvature tensor

T_{ab} - stress-energy tensor

S^n - the unit n sphere

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Chapter 1

Introduction

Over the past several decades, it has been of interest to understand and model highly compact astrophysical systems in general relativity. This process is significantly more complicated than modeling the same systems in Newtonian gravity.

In full general relativity, one is required to solve ten nonlinear coupled partial differential equations (Einstein's equations) to determine a Lorentzian metric on a topological manifold. The metric determines most of the interesting properties of the manifold such as its geodesics and curvature. It is well known that general relativity predicts many gravitational phenomena that have been experimentally observed which are absent in Newton's theory. Canonical examples are the precession of the perihelion of Mercury, gravitational waves, and black holes.

Black holes and other compact astrophysical objects have been a mainstream research topic since Schwarzschild's original derivation of the spacetime exterior to a spherical point source in general relativity. Since his day, we have come to understand that not only do variations of such highly symmetric strong gravity solutions exist in large numbers in the universe. Kerr later found a generalization of the Schwarzschild solution which includes potential rotation. Further work established that the Kerr spacetime is the unique stationary axisymmetric rotating solution in general relativity. It is interpreted as the endstate of most gravitational collapse processes. While this is a startling result and begs astronomers to find such objects, it seems unlikely black holes will exist in a form as pure as the Kerr solution. One expects in the vicinity of a black hole that there will be at a minimum a interstellar medium. Additional matter such as winds, other stars, or accretion disks may also be present. In short, one expects to find "dirty black holes" and not the mathematical result of Kerr. So an interesting question becomes, what are the interactions of a black hole with more complicated astrophysical systems? One can further speculate because of more recent theories from particle physics that one must extend the Kerr solutio to include other matter fields and higher dimensions. There are a host of questions that go beyond the original Kerr solution into considerations of time dependent higher dimensional black holes.

The overarching themes of this thesis are such considerations. Because general relativity is so complex, it is difficult to construct exact solutions of the theory. This naturally becomes even worse if one adds higher dimensions or complicated matter fields. To have a reference point as we mentioned, the first solution to the vacuum Einstein equations was due to Schwarzschild and is a Lorentzian manifold (M, g) where the metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (1.1)$$

By inspection of the metric, one can see that degeneracies arise when $r = 0$ and $r = 2M$. Further analysis shows that $r = 2M$ is a coordinate singularity and is thus not physical. The $r = 0$ singularity is a curvature singularity where the spacetime breaks down. This was the first theoretical evidence of the existence of black holes. Moreover, Birkhoff showed that the Schwarzschild solution is the unique spherically symmetric solution of the Einstein equations. One might therefore expect that if we perturb a Schwarzschild spacetime in a spherically symmetric manner, that the resulting spacetime will return to the original. Said another way, one might assume the original spacetime is stable. This was indeed shown to be true in [29]. One might expect other solutions such as Kerr to be stable. In general answering such a question is very difficult because it reacquires solving the full Einstein equations. We instead can use linear perturbation theory to resolve such questions to linear order. Chapter 3 is concerned with such problems.

The Einstein equations admit more than just spherically symmetric of axisymmetric vacuum solutions. If we source the Einstein tensor with the Maxwell stress energy tensor, we can generate a spherically symmetric solution of the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.2}$$

which reduces to the Schwarzschild solution when $q = 0$. This is called the Reissner-Nordstrom metric and is the simplest metric that incorporates electromagnetism into general relativity via a charge parameter q. One can also source the gravitational field with more general stress tensors. One could consider a fluid sourcing the Einstein equations. Indeed there are spherically symmetric solutions for certain types of fluids representing both static stars and spherically symmetric gravitational collapse. One could imagine extending these to include the Maxwell tensor and assuming the matter content of magnetohydrodynamics. In this case we can no longer construct exact solutions to the Einstein equations. Because of the resulting complexity of the equations, we must then consider numerical techniques that arise when solving the Einstein equations. One expects the universe to be populated by compact magnetized fluid objects which need descriptions by general relativistic magnetohydrodynamics. Such objects might include, neutron stars, magnetized white dwarfs, pulsars, magnetars, gamma-ray bursts, supernovas, and a variety of binary or multiple star systems. Chapters 4 and 5 are concerned with developing models for these objects.

In Chapter 2 we give a brief survey of Einstein's original formulation of general relativity. Variations of our exposition can be found in all canonical relativity texts listed in the chapter. We will see that Einstein's original formulation is physically motivated but not easily generalizable. We later wish to add an auxiliary dilaton field to the field equations of relativity. To this end, we give a modern derivation of the Einstein field equations in terms of extremizing the "simplest" functional of the curvature of a metric defined on a Lorentzian manifold. We use standard techniques in geometric analysis to perform this operation rather than analogous methods found in standard physics texts. This method will allow us to easily extend to more general theories of gravity in the subsequent chapter by adding appropriate fields to our curvature functional.

In Chapter 3 we consider stability analyses of black strings and p-branes. Black strings are solutions to the vacuum Einstein equations in $D > 4$ dimensional spacetimes that reduce to the Schwarzschild solution when $D = 4$. The bulk of our work consists of fixing a solution to a given theory of gravity, perturbing the solution, and analyzing the growth or decay of the perturbation in time. If the perturbations are not bounded in time, we conclude the original black object was unstable. We first summarize a classical stability problem due to Jeans that motivates our discussion of black string instability. Black strings are one possible extension of the Schwarzschild solution to higher dimensions. It is a rather surprising fact that we find black strings are linearly unstable whereas the Schwarzschild solution is linearly stable. We develop the full analytical and numeric analysis necessary to show black strings are unstable. This involves computing the Einstein equations to linear order for a perturbation metric. We then combine these equations to a single equation for one component of the perturbation. Numerical integration of this equation shows that the perturbation will grow exponentially large as time increases.

We then consider charged black p-brane solutions of low energy string theory. These objects may be heuristically thought of as magnetically charged black strings. We briefly recall the seminal Gregory-Laflamme analysis [11] of a class of charged black p-branes that were concluded to be unstable up to the extremal limit of charge. In the extremal limit, Gregory-Laflamme showed the instabilities in the finitely charged black p-branes disappeared. Their work is similar in form to a more complicated version of our black string considerations.

We finally discuss the stability of a more general class of p-branes that was originally considered in a paper by Reall [26]. We extend the numerical analysis of [19] to reproduce all the results of Gregory/Laflamme [11, 12], and show there exists a wide class of p-brane solutions where linear instabilities vanish that are determined by a constant coupling the dilaton field to a generalized generalized Maxwell field. We plot our results and present conclusions.

Chapter 4 is primarily concerned with posing and solving the eigenvalue problem of general relativistic magnetohydrodynamics (GRMHD). GRMHD represents the coupling of Einstein's equations to electrodynamics and fluid dynamics in the ideal MHD approximation. It is extremely complicated and highly nonlinear. Our motivation for studying this problem is due largely to potential numerical benefits concerning the simulation of the GRMHD equations.

In [4] Brio and Wu construct a Roe solver for non-relativistic magnetohydrodynamics. Roe solvers are characterized by their robustness and ability to resolve shocks (discontinuities). The solver requires one to represent the MHD equations in conservative form, and to compute the eigenvalues and eigenvectors of the associated Jacobian matrix for all spacial steps. Moreover, the eigenvectors must form a complete set for the solver to work properly.

We show that the MHD equations do not initially admit a complete set of eigenvectors. There are certain parameter values which cause eigenvalues and eigenvectors of MHD to degenerate or become singular. We find all such singularities and scale the Jacobian matrix by the appropriate factor to guarantee existence of a complete set of eigenvectors. The resulting MHD equations are then ready for numerical integration via a Roe solver.

We next extend the MHD methods to GRMHD. We give a formulation of the GRMHD equations due to [20] and calculate the eigenvalues of their Jacobian matrix. There are seven nontrivial eigenvalues that are significantly more complicated in form than their MHD analogous. Four of the eigenvalues are given as solutions to a non-factorizable quartic that have no simple algebraic representations. We compute the associated right eigenvectors and give degeneracy and singularity conditions on both. These results will be used to eliminate spurious waves originating from the boundary of the numerical grid of current GRMHD numerical simulations.

Spherical symmetry is too restrictive for modeling problems in GRMHD. We consider axisymmetric spacetimes instead. This assumption makes the resulting Einstein equations slightly more complicated than those in the spherically symmetric case. Solving the axisymmetric Einstein equations coupled to the GRMHD stress tensor allows us to model magnetized neutron stars and charged black holes.

In Chapter 5, we develop equations for the initial value problem of a magnetized neutron star in the context of general relativity. One motivation for this work is to model gravitational radiation emanating from the star or its interaction with other astrophysical objects. Direct detection of gravitational waves is one of the major open problems in experimental physics. The numerical simulation of magnetized neutron stars will potentially admit gravitational waveforms and hence allow experimentalists to know what type of waves to look for.

If one wishes to evolve an initial spacetime in time according to the Einstein equations, one may not choose the initial spacetime arbitrarily, but must satisfy a set of constraint equations. This is analogous to choosing initial data in electrodynamics consistent with the equations of electrostatics. The initial value problem for general relativity is itself highly nontrivial, often resulting in solving a system of coupled second order partial differential equations.

We first define the notion of a circular axisymmetric spacetime. Matter may only propagate in planes perpendicular to an axis of symmetry in such a spacetime. If one assumes a circular spacetime, the metric and resulting Einstein equations simplify dramatically. We then consider the work of Cook et. al. in [5] where a circular axisymmetric initial value problem is considered for general relativistic hydrodynamics. We derive the Einstein and matter equations associated with this model. Moreover, we briefly comment on an iterated Green's function technique that can be used to numerically solve the Einstein equations. This method may also be used to solve the similar Maxwell equation in our later GRMHD analysis.

We finally proceed to formulate the equations for non-circular axisymmetric GRMHD. We do this by decomposing a general Lorentzian metric along spacelike and timelike Killing vectors and by calculating the Einstein equations, matter equations, and Maxwell equations. Our equations are relatively aesthetic as compared to [3] where non-differentially rotating magnetized neutron stars were considered. We leave numerical computations for later work.

We finish with a summary of our conclusions in Chapter 6.

Chapter 2

Einstein's Field Equations

To begin our investigation of time dependent solutions in gravity, we provide a brief review of the Einstein equations in order to set the stage for our efforts to solve them (See [6, 17, 18, 30, 31] more information on the Einstein equations). We then provide a variational formulation for the purpose of motivating extensions of general relativity to more general gravitational theories. This method can also be used to establish the equations of motion for low energy string theory.

The Einstein equations serve as the governing equations of general relativity and are hence fundamental to all topics discussed in this thesis. We note that our variational formulation is based on standard techniques in geometric analysis rather than canonical relativity texts. For an introduction to standard geometric analysis methods see [25, 28].

2.1 Original Formulation

Einstein first formulated his field equations of general relativity by building an extended analogy between constructs in Newtonian gravity and Lorentzian geometry.

In Newtonian gravity, the gravitational field is defined in terms of a scalar potential ϕ. If we place two test particles initially separated by a vector x in a gravitational field generated by a spherically symmetric mass, the particles will accelerate towards the mass. Moreover, they experience tidal acceleration given by $-(x \cdot \nabla)\nabla\phi$.

In general relativity, we model gravity as a Lorentzian manifold of topology M and metric g. We then define a covariant differentiation operator, which computes derivatives of tensors and projects the resulting expression into the tangent plane of the manifold pointwise. There is a unique way to perform this operation such that the metric is parallel transported along any curve in the manifold. We call the connection that builds such a covariant derivative the Levi-Civita connection. With this connection, we may consider the quasi-linear system of equations whose solutions are the geodesics of (M, g).

Geodesics locally extremize the length functional between two points on a Lorentzian manifold. Moreover, this property is global if the manifold topology is sufficiently well behaved. In general relativity, gravity is no longer a force but a consequence of the curvature of a spacetime defined by a metric tensor. Free particles should travel along extremal paths in this spacetime. It is thus natural to hypothesize that test particles travel along geodesics of spacetime.

The geodesic deviation equation states that two geodesics "test particle trajectories" in close proximity have tidal acceleration $-R_{cbd}{}^{a}v^{c}x^{b}v^{d}$ where v^a is the four velocity of the geodesics (particles) and x^a is the separation vector of the particles. Heuristically, this suggests an identification with Newtonian gravity

$$R_{cbd}{}^{a}v^{c}v^{d} \sim \partial_b\partial^a\phi \tag{2.1}$$

where partial derivatives with free indices a and b have replaced the previous gradient notation. In the presence of a matter distribution of mass density ρ, the defining

Newtonian relation for the gravitational potential is

$$\nabla^2\phi = \partial_a\partial^a\phi = 4\pi\rho. \tag{2.2}$$

In general relativity, matter will no longer be represented by a scalar quantity. Instead, we encode all information about matter in terms of a quantity we call the stress energy tensor T_{ab}. The stress energy tensor contains all information about energy density, linear momentum, and stresses of fields. Specifically, its time components contain the energy of a given matter field. Since $v^a = (1, 0, 0, 0)$ in the frame of an observer, we may pick off the tt-component of T_{ab} and identify this tensor with ρ via

$$T_{ab}v^a v^b \sim \rho \tag{2.3}$$

noting that the contractions are necessary to make the left hand side a scalar quantity. By analogy with Poisson's equation this yields

$$R_{cad}{}^{a} v^c v^d \sim T_{cd}v^c v^d \rightarrow R_{cad}{}^{a} = \kappa T_{cd} \tag{2.4}$$

where we have chosen a coupling constant, κ, which will be calculated from the weak field Newtonian limit.

This was Einstein's first attempt to formulate general relativity as the geometrical analogue of Newtonian gravity. However, one can show that $\nabla_a T^{ab} = 0$ is the relativistic analogue of the classical principle of conservation of energy, and in general $\nabla_a R^a{}_{cb}{}^c \neq 0$. Thus (2.4) violates conservation of energy.

This problem is overcome by adding a term to the previous equation to get

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}. \tag{2.5}$$

Equations (2.5) are the field equations of general relativity and are called Einstein's equations. They express a nonlinear relationship between the curvature of a spacetime manifold (gravity) and the stress energy tensor (matter).

The preceding argument relies heavily on an analogy with Newtonian gravity. The reason general relativity is a successful theory lies in the fact that equations (2.5) have predicted observed phenomena such as the precession of the perihelion of Mercury, the existence of black holes, and cosmological expansion, which are not consequences of Newtonian gravity. Since it can be shown that general relativity reproduces Newtonian gravity in the low energy limit (weak gravitational fields), relativity is viewed as a more accurate theory of gravity.

2.2 Alternative Variational Formulation

In this section we demonstrate how Einstein's equations may be derived in a mathematically rigorous way from a variational principle. This derivation is devoid of a classical analogy and thus must be taken in tandem with the above to suggest a viable theory of gravity.

Consider a Lorentzian manifold (M, g). Let h be another metric on M and $\epsilon \in \mathbb{R}^+$. Let $S[g]$ be any functional of g. Then for g_{ab} and h_{ab}, the components of g and h respectively, we define the first variation of g with respect to h by

$$\delta_h(S) \equiv \frac{d}{d\epsilon} S[g_{ab} + \epsilon h_{ab}]\bigg|_{\epsilon=0} \tag{2.6}$$

We will also consider the first variation of metric dependent differential operators and tensors. We require the following lemma to derive Einstein's equations:

Lemma 1. *Let (M, g) be a Lorentzian manifold, $R(g)$ the scalar curvature of g, and $Ric(g)$ the Ricci curvature of g. Then*

$$\delta_h R(g) = -\mathrm{tr}_h(Ric(g)) - \Delta(\mathrm{tr}_g h) + \mathrm{div}^2 h \tag{2.7}$$

$$= -h^{ij} R_{ij} - \nabla_i \nabla^i (h^{jk} g_{jk}) + \nabla_i \nabla_j h^{ij} \tag{2.8}$$

where in local coordinates tr *is the trace operator on tensors with two covariant indices with respect to a metric h. For example $tr_g h = g^{ij} h_{ij}$.*

Proof: We first note that $\delta_h(g_{ab}) = h_{ab}$ is immediate from (2.6) taking S to be the identity functional. Also from the formula $g_{ac}g^{cb} = \delta_a^b$ and noting the Leibnitz property holds for δ_h, we compute $\delta_h g^{ab} = -h^{ab}$. We fix a $p \in M$ and choose geodesic coordinates at p. In these coordinates $g_{ab}|_p = \eta_{ab}$, $\partial_k g_{ab}|_p = 0$, and $\Gamma^a_{bc}|_p = 0$, which allows us to compute

$$\delta_h \Gamma^a_{bc} = \delta_h \left(\frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right) \right) \tag{2.9}$$

$$= \frac{1}{2} g^{ad} \delta_h \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right) - \frac{1}{2} h^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right) \tag{2.10}$$

$$= \frac{1}{2} g^{ad} \left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc} \right) \tag{2.11}$$

$$= \frac{1}{2} g^{ad} \Big(\left[\partial_b h_{cd} - h_{ed}\Gamma^e_{bc} - h_{ce}\Gamma^e_{bd} \right] + \left[\partial_c h_{bd} - h_{ed}\Gamma^e_{bc} - h_{be}\Gamma^e_{cd} \right] \tag{2.12}$$

$$- \left[\partial_d h_{bc} - h_{ec}\Gamma^e_{bd} - h_{be}\Gamma^e_{cd} \right] \Big) \tag{2.13}$$

$$= \frac{1}{2} g^{ad} \left(\nabla_b h_{cd} + \nabla_c h_{bd} - \nabla_d h_{cd} \right). \tag{2.14}$$

Similarly one can show

$$\delta_h R^a{}_{bcd} = \frac{1}{2} g^{ae} \left(\nabla_c \nabla_d h_{be} - \nabla_c \nabla_e h_{bd} - \nabla_b \nabla_e h_{cd} + \nabla_c \nabla_b h_{de} - \nabla_b \nabla_c h_{de} \right) \quad (2.15)$$

$$\delta_h R_{ab} = -\frac{1}{2}\Delta_L h_{ab} - \frac{1}{2}\nabla_a \nabla_b \mathrm{tr}_g h + \nabla_b (\mathrm{div} h)_a + \nabla_a (\mathrm{div} h)_b \quad (2.16)$$

$$= -\frac{1}{2}\Delta_L h_{ab} - \frac{1}{2}\Delta(\mathrm{tr}_g h) - \mathrm{div}^* \mathrm{div} h \quad (2.17)$$

$$\delta_h R = -h^{ab} R_{ab} - \Delta(\mathrm{tr}_g h) + \mathrm{div}^2 h \quad (2.18)$$

where we define the Lichnerowicz Laplacian Δ_L on covariant two tensors by[1]

$$\Delta_L T_{ab} \equiv \Delta T_{ab} + 2 g^{cd} R^e{}_{cab} T_{ed} - g^{cd} R_{ad} h_{cb} - g^{cd} R_{bd} h_{ac}. \quad (2.19)$$

We also define div as contraction of its argument with a covariant derivative. For example $\nabla_a (\mathrm{div} h)_a = \mathrm{div}(\mathrm{div}(h)) = \mathrm{div}^2 h$. For a one form ω with components ω_i we define the adjoint divergence operator

$$\mathrm{div}^* \eta \equiv -\frac{1}{2}(\nabla_a \eta_b + \nabla_b \eta_a) \quad (2.20)$$

where * indicates the adjoint operator. The last tool we require for the derivation of the Einstein equations is the following

Lemma 2. *Let (M, g) be a Lorentzian manifold and $d\mu$ its volume form. Then*

$$\delta_h (d\mu) = \frac{1}{2}(\mathrm{tr}_g h) d\mu \quad (2.21)$$

.

Proof: For two matrices a_{ij}, b_{ij}, we note [25]

$$\frac{d}{d\epsilon} \det(a_{ij} + \epsilon b_{ij}) \bigg|_{\epsilon = 0} = a^{ij} b_{ij} \det(a_{ij}) \quad (2.22)$$

[1] Recall the standard Laplacian is given by $\Delta T_{ab} = g^{cd} \nabla_c \nabla_d T_{ab}$.

and compute

$$\delta_h(d\mu) = \delta_h\left(\sqrt{|g|}\right)dx = \frac{\delta_h \det(g)}{2\sqrt{\det(g)}}dx = \frac{1}{2}(\mathrm{tr}_g h)\sqrt{|g|}dx = \frac{1}{2}(\mathrm{tr}_g h)d\mu. \tag{2.23}$$

We now have the necessary machinery to define and derive the Einstein equations. For a Lorentzian manifold (M, g), we define the vacuum Einstein equations to be the system of equations given by demanding the first variation of

$$S_E(g) = \int_M R d\mu \tag{2.24}$$

is trivial. We explicitly compute the first variation of this action

$$\delta_h S_E = \int_M [R(\delta_h d\mu) + (d\mu)\delta_h R] = \int_M \left[-h^{ab}R_{ab} - \Delta \mathrm{tr}_g h + \mathrm{div}^2 h + \frac{R}{2}\mathrm{tr}_g h\right] d\mu \tag{2.25}$$

$$= -\int_M h^{ab}\left[R_{ab} - \frac{1}{2}R g_{ab}\right] d\mu \tag{2.26}$$

Thus we see that $\delta_h S_E = 0$ iff $G_{ab} \equiv R_{ab} - \frac{1}{2}R g_{ab} = 0$. Thus the vacuum Einstein equations are $G_{ab} = 0$.

2.3 Matter Einstein Equations

We often wish to source the gravitational field with a mass/energy distribution. To include this in our variational formalism, we augment the general relativity action via

$$S = S_E + S_M \quad \text{where} \quad \delta_h(S_M) = \int_M 8\pi h^{ab} T_{ab} d\mu. \tag{2.27}$$

This reproduces the matter Einstein equations

$$G_{ab} = 8\pi T_{ab} \tag{2.28}$$

where the coefficient is chosen to agree with Newtonian gravity in the weak field limit. We now turn to deriving and solving these equations for various stress energy tensors.

Chapter 3

Black String Stability

In this chapter we consider questions associated with the stability of black objects in general relativity and its extension to higher dimensions. We begin with a simple example taken from Newtonian gravity to motivate more difficult problems in general relativity and low energy string theory.

Stability investigations of solutions to Einstein's field equations and their analogues in low energy string theory have been widely treated in the physics literature. For a recent review, see [16]. The problem of linear stability of a spacetime can be heuristically summarized by the following: Suppose one has a Lorentzian manifold $(M, \widetilde{g})$ where $\widetilde{g}$ is a solution of prescribed field equations. Let h be another metric on M and define a perturbed metric $g = \widetilde{g} + \epsilon h$ for $0 < \epsilon << 1$. One may use the field equations to linear order in ϵ for g to solve for the perturbation h. If it can be concluded that h decays as time goes to infinity, one says the original spacetime $(M, \widetilde{g})$ is linearly stable; if not it is unstable. We will refer to $\widetilde{g}$ as the fixed background metric and to h as the perturbation metric. It is interesting to classify spacetimes according to their stability for purposes of discussing their physicality. If a spacetime is linearly unstable it is unlikely that it is a physical model; however, this does not preclude the remote possibility that nonlinear effects could potentially stabilize a linearly unstable spacetime.

3.1 Introduction

Linear stability analysis in gravity originated with the work of Jeans [23] at the turn of the twentieth century. He considers a gravitational perturbation of a static matter distribution with constant mass density $\rho = \rho_0$ and constant pressure $p = p_0$. The governing equations for the system are the equations of hydrodynamics[1].

$$\partial_t \rho + \nabla \cdot (\rho \, \mathbf{v}) = 0 \tag{3.1}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{g} \tag{3.2}$$

and the gravitational field equations

$$\nabla \times \mathbf{g} = 0 \tag{3.3}$$

$$\nabla \cdot \mathbf{g} = -4\pi G \rho \tag{3.4}$$

where $\mathbf{v}$ is the fluid velocity and $\mathbf{g}$ is the gravitational force field. If we now consider $p_0 + \widetilde{p}$ with $\widetilde{p}$ a small perturbation of the pressure and analogous perturbations for the other variables, we find that the governing equations to linear order in the perturbation variables are

$$\partial_t \widetilde{\rho} + \rho_0 \nabla \cdot \widetilde{v} = 0, \quad \partial_t \widetilde{v} = -\frac{1}{\rho_0} \nabla \widetilde{p} + \widetilde{\mathbf{g}} \tag{3.5}$$

[1] We typically choose units where $G = 1$; however, we leave G in standard units for this example.

$$\nabla \times \widetilde{\mathbf{g}} = 0, \quad \nabla \cdot \widetilde{\mathbf{g}} = -4\pi G\widetilde{\rho}. \tag{3.6}$$

These may be combined to yield the equation

$$\partial_t^2 \widetilde{\rho} = \Delta \widetilde{p} + 4\pi G \rho_0 \widetilde{\rho}. \tag{3.7}$$

Next we assume an equation of state of the form $\widetilde{p} = v_s^2 \widetilde{\rho}$ where v_s is the speed of sound, make a Fourier ansatz $\widetilde{\rho} = Ae^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$, and substitute into (3.7) yielding

$$\omega^2 = v_s^2 k^2 - 4\pi G \rho_0. \tag{3.8}$$

Note that if the imaginary part of ω is negative, then $\widetilde{\rho}$ will increase exponentially in time, which indeed happens for long wavelengths

$$\lambda > \lambda_* \equiv \sqrt{\frac{\pi v_s^2}{G\rho_0}}. \tag{3.9}$$

Thus an object is unstable to gravitational perturbation if its wavelength is larger than λ_*.

One can perform similar but more complicated stability analyses in general relativity. In [29] the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{3.10}$$

is shown to be linearly stable. We will first perform a similar analysis for Schwarzschild type solutions of the Einstein equations in higher dimensions. These solutions are often referred to as black strings. Analyzing the linear stability of a spacetime allows us to make statements about the possible physical realization of a spacetime. For instance, if a spacetime is unstable, then it is unlikely to be physically realized. Stability analyses of solutions in higher dimensional relativity are motivated by string theory and serve as examples of more complicated models that arise in higher dimensional theories of gravity.

3.2 Black Strings

Black string instabilities were first discussed by Gregory and Laflamme (GL) in [10]. There have been several follow ups of the original analysis. Our work is similar in content to that of [22] but differs in our choice of numerical method. We calculate the parameters related to the occurrence of instabilities to higher accuracy and are in agreement with both [10] and [22].

To begin, we define a black string in $D = n + 4$ dimensions to be a Lorentzian manifold $(M, \widetilde{g})$ with metric

$$ds^2 = \widetilde{g}_{ij} dx^i dx^j = -f dt^2 + f^{-1} dr^2 + dz^2 + r^2 d\Omega^2_{n+1}, \quad f = 1 - \left(\frac{r_+}{r}\right)^n \tag{3.11}$$

where $d\Omega^2_{n+1}$ is the canonical metric on S^{n+1}, z is a single compactified extra dimension, and r_+ is a constant determining the event horizon of the black string. We note that the form of the black string metric is a generalization of the Schwarzschild metric (3.10). Recall the event horizon of a Schwarzschild solution has a spherical horizon topology. A black string with one extra compact dimension has horizon topology $S^{n+1} \times S^1$, and thus can be thought of as a circle (string) with S^{n+1} cross sections.

Now let h be any symmetric bilinear form on M. For a fixed metric $\widetilde{g}$ we define its perturbation by h to be

$$g_{\mu\nu} = \widetilde{g}_{\mu\nu} + \epsilon h_{\mu\nu} \tag{3.12}$$

where $0 < \epsilon << 1$. We set $h = g^{\mu\nu}h_{\mu\nu}$, and recall from [30, p.185] that the linearized Einstein equations for $g_{\mu\nu}$ to linear order in ϵ are given by

$$0 = \nabla_a \nabla_c h + \nabla^b \nabla_b h_{ac} - 2\nabla^b \nabla_{(c} h_{a)b} \tag{3.13}$$

$$= \nabla_a \nabla_c h + \nabla^b \nabla_b h_{ac} - \nabla^b \nabla_c h_{ab} - \nabla^b \nabla_a h_{cb} \tag{3.14}$$

where all covariant derivatives are defined with respect to the background metric, $\widetilde{g}$.

We wish to simplify these equations by choosing appropriate coordinates. General relativity is a tensorial theory in all dimensions and thus admits the diffeomorphism group as a symmetry group. Thus we are free to choose local coordinates which is commonly called gauge fixing. We will fix our gauge to simplify the form of the perturbation as much as possible, which we know illustrate.

In [11] the authors consider a local coordinate transformation

$$x^a \to x^a + \hat{\xi}^a(x^a) \tag{3.15}$$

where $\hat{\xi}^a$ is infinitesimal. The corresponding perturbation transformation is

$$h^{ab} \to h^{ab} + \nabla^{(a}\hat{\xi}^{b)} \tag{3.16}$$

which is shown in [30, p.75]. We demand that the perturbation takes a spherically symmetric form. This implies that $\hat{\xi}^\theta = \hat{\xi}^\phi = 0$, i.e. the angular components of our perturbation vanish and nonzero components are independent of the angular coordinates. We choose such a perturbation since it will simplify our the form of our perturbation and because our background spacetime is spherically symmetric. Specifically, our perturbation now takes the form

$$h_{ab} = e^{\Omega t + ikz} a_{ab}(r) \equiv \Xi a_{ab}, \quad \hat{\xi}^a = \xi^a(r)\Xi \tag{3.17}$$

where we have decomposed h_{ab} into Fourier modes with Ω in the time direction and k along the extra S^1. We are only concerned with Ω being real since positive Ω is sufficient to imply instability. We only need to find a single positive real Ω to show instability. Demanding spherical symmetry requires $a_{ab}(r)$ and $a_{\theta_i z} = 0$ where θ_i is any angular coordinate. Using our assumption of spherically symmetry, the perturbation metric transforms to

$$h_{ab} \rightarrow h_{ab} + \frac{1}{2}\left[\nabla_a \hat{\xi}_b + \nabla_b \hat{\xi}_a\right] \tag{3.18}$$

$$= \Xi a_{ab} + \frac{1}{2}\left[\partial_a \hat{\xi}_b + \partial_b \hat{\xi}_a - 2\Gamma^c_{ab}\hat{\xi}_c\right] \tag{3.19}$$

$$= \Xi a_{ab} + \frac{1}{2}\left[\partial_a \hat{\xi}_b + \partial_b \hat{\xi}_a - \hat{\xi}^d(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})\right]. \tag{3.20}$$

We thus compute the following transformed perturbation metric components

$$h_{tt} \rightarrow \Xi\left[a_{tt} + \Omega\xi_t - \frac{1}{2}r_+^n r^{-n-1}\xi^r\right] \tag{3.21}$$

$$h_{tr} \rightarrow \Xi\left[a_{tr} + \frac{1}{2}(\Omega\xi_r + \partial_r\xi_t + \xi^t(r_+^n n r^{-n-1}))\right] \tag{3.22}$$

$$h_{tz} \rightarrow \Xi\left[a_{tz} + \frac{1}{2}\left(\Omega\xi_z + ik\xi_t\right)\right] \tag{3.23}$$

$$h_{t\theta} \rightarrow 0 \tag{3.24}$$

$$h_{rr} \rightarrow \Xi\left[a_{rr} + \partial_r\xi_r + \xi^r\frac{nr^{n-1}r_+^n}{2(r^n - r_+^n)^2}\right] \tag{3.25}$$

$$h_{rz} \rightarrow \Xi\left[a_{rz} + \frac{1}{2}(\partial_r\xi_z + ik\xi_r)\right] \tag{3.26}$$

$$h_{zz} \rightarrow \Xi\left[a_{zz} + ik\xi_z\right] \tag{3.27}$$

$$h_{\theta\theta} \rightarrow \Xi\left[a_{\theta\theta} + \frac{1}{2}\xi^r\partial_r g_{\theta\theta}\right] \tag{3.28}$$

We have freedom to choose ξ^a however we like. We choose to force the perturbation to be in the simplest form possible. Specifically, we choose

$$a_{tt} = h_t(r), \quad a_{rr} = h_r(r), \quad a_{zz} = h_z(r), \quad a_{tr} = -\frac{\Omega}{ik} a_{zr}(r), \quad a_{tz} = a_{\theta\theta} = 0. \tag{3.29}$$

Note in the first three expressions we have merely renamed variables, while the last three are bona-fide gauge fixing (coordinate choices). These choices result in

1. $a_{\theta\theta} = 0$ iff $\xi_r = -2a_{\theta\theta}/(g^{rr}\partial_r g_{\theta\theta})$

2. $a_{tz} = 0$ iff $\Omega\xi_z + ik\xi_t = -2a_{tz}$

3. $a_{tr} = -\frac{\Omega}{ik}a_{zr}$ iff $\partial_r\xi_t + \xi^t n r_+^n r^{-n-1} + \frac{i\Omega}{k}\partial_r\xi_z = 2a_{tr} - \frac{2i\Omega}{k}a_{rz}$.

Thus condition (1) fixes ξ_r, while conditions (2) and (3) determine a linear system of first order ordinary differential equations that fix the remaining gauge components ξ_t, ξ_z. In this way, we use all our available gauge freedom. The perturbation is of the form

$$h_{\mu\nu} = \Xi \begin{pmatrix} h_t & \Omega h_\nu & 0 & 0 & \cdots \\ \Omega h_\nu & h_r & \frac{k}{i}h_\nu & 0 & \cdots \\ 0 & \frac{k}{i}h_\nu & h_z & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{3.30}$$

where the h_i are only r dependent and $h_\nu = a_{tr}/\Omega$. We note that in the original analysis of [10] transverse traceless (TT) gauge was chosen to simplify the perturbation equations rather than the form of the perturbation. The gauge choice that we have taken leaves the perturbation equations in a more complicated form; however, it simplifies the form of the perturbation [22]. In comparison to TT gauge, this choice has the advantage of leaving no residual gauge freedom as well as simplifying the construction of a differential equation for the h_z perturbation component.

To determine the perturbation equations, we derive the linearized Einstein equations by computing the Ricci tensor for the perturbation metric, defining a first order linearization operator in the perturbation parameter ϵ, and linearizing the Ricci tensor. Combining components of the Ricci tensor yields a linear system for h_i, h_i', h_i'' which is completely determined by a single equation in h_z

$$h_z'' + ph_z' + qh_z = \Omega^2 w h_z \tag{3.31}$$

where

$$p = \frac{1}{r}\left(1 + \frac{n}{f(r)} - \frac{4(n+2)k^2r^2}{2k^2r^2 + n(n+1)(r_+/r)^n}\right) \tag{3.32}$$

$$q = -\frac{1}{r^2}\left(\frac{k^2r^2}{f(r)}\frac{2k^2r^2 - n(n+3)(r_+/r)^n}{2k^2r^2 + n(n+1)(r_+/r)^n}\right), \quad w = \frac{1}{f(r)^2} \tag{3.33}$$

This is a significant simplification when compared to the corresponding result in [11].

3.3 Asymptotic Behavior of the main Perturbation Equation

We need to know how (3.31) behaves near the horizon r_+ and near spacial infinity so we may determine appropriate boundary conditions for numerical simulation. We will require h_z to vanish at spacial infinity if it is to be a candidate for a physical solution. This will be made manifest in our asymptotic boundary conditions.

3.3.1 Behavior at the Horizon, $r = r_+$

In order to analyze the near horizon behavior of (3.31), we define a parameter $\epsilon = r - r_+$, and note that

$$f = 1 - \frac{r_+^2}{r^2} = \frac{2}{r_+}\epsilon + O(\epsilon^2). \tag{3.34}$$

We find that p and w in equation (3.31) to lowest order in ϵ are

$$p \rightarrow \frac{1}{\epsilon}, \quad w \rightarrow \frac{\Omega^2 r_+^2}{4\epsilon^2}. \tag{3.35}$$

Thus we find the near horizon equation in ϵ to be

$$h_z'' + \frac{1}{\epsilon}h_z' = \frac{\Omega^2 r_+^2}{4\epsilon^2}h_z \tag{3.36}$$

with solution

$$\begin{aligned} h_z &= c_1\epsilon^{\frac{r_+\Omega}{2}} + c_2\epsilon^{\frac{-r_+\Omega}{2}} && (3.37)\\ &= c_1(r - r_+)^{\frac{r_+\Omega}{2}} + c_2(r - r_+)^{-\frac{r_+\Omega}{2}} && (3.38)\end{aligned}$$

which agrees with the result in [10]. The near horizon solution can also be written [22]

$$h_z = Ae^{\Omega r_*} + Be^{-\Omega r_*} \tag{3.39}$$

where r_* is the tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{1}{f} \rightarrow r_* = r_+ \mathrm{arctanh}\left(\frac{r}{r_+}\right). \tag{3.40}$$

We note this asymptotic behavior is in agreement with [11] since

$$h_z = A_1(r - r_+)^{\frac{\Omega r_+}{2}}(r + r_+)^{-\frac{\Omega r_+}{2}} + A_2(r - r_+)^{-\frac{\Omega r_+}{2}}(r + r_+)^{\frac{\Omega r_+}{2}}$$

$$\sim \widetilde{A}_1(r-r_+)^{\frac{\Omega r_+}{2}} + \widetilde{A}_2(r-r_+)^{-\frac{\Omega r_+}{2}} \tag{3.41}$$

because $r + r_+$ is non-singular near the horizon.

3.3.2 Behavior as $r \to \infty$

To find the asymptotic behavior of (3.31) we let $r \to 1/s$ and investigate the limit $s \to 0$. We compute transformed derivatives

$$\partial_r h_z = -s^2 \partial_s h_z \quad \partial_r^2 h_z = s^4 \partial_s^2 h_z + 2s^2 \partial_s h_z \tag{3.42}$$

and note that $f(r) = f(1/s) = 1 - r_+^2 s^2$. We again compute the lowest order terms of (3.31) in each of the coefficients, and find the asymptotic equation

$$h_z'' + \frac{7}{s} h_z' - \frac{\mu^2}{s^4} h_z = 0 \tag{3.43}$$

where we have defined $\mu^2 = \Omega^2 + k^2$. This is a modified Bessel equation. We only keep the K_ν part of the solution since the I_ν become infinite in the asymptotic regime. This will become a physical boundary condition. Solving and inverting the coordinate transformation, we find [2]

$$h_z = c_3 r^3 K_3(r\mu) \sim c_3 r^3 \left(\frac{e^{-r\mu}}{r^{1/2}} \right) = c_3 r^{5/2} e^{-r\mu} \tag{3.44}$$

which is also found in [22]. Note that this decays exponentially for large r.

3.4 The Shooting Method

We will use the shooting method to numerically solve the eigenvalue problem (3.31). For an eigenvalue problem with known boundary conditions, we set the eigenvalue in the differential equation to some fixed value which may or may not be a true eigenvalue of the equation. We then integrate in r outward from the horizon. If the asymptotic boundary condition is satisfied by our numerical solution, we have found a true eigenvalue. If not, we vary our eigenvalue choice until the boundary conditions are satisfied. We first give a simple example of this method.

3.4.1 Example

Our ultimate aim is to solve equation (3.31) numerically by the shooting method. For that purpose we will summarize how to code the integration method in Matlab for a harmonic oscillator eigenvalue problem.

To fix ideas, we consider a harmonic oscillator problem

$$\ddot{y}_2(t) + \Omega y_2(t) = 0 \tag{3.45}$$

$$y_2(0) = 1 \quad \dot{y}_2(0) = 0. \tag{3.46}$$

We could solve this problem exactly considering Ω as an arbitrary parameter. This becomes an eigenvalue problem if in addition to the initial conditions we impose a boundary condition. This may be decomposed into the following system of first order equations

$$\dot{y}_1 = -\Omega y_2, \quad y_1(0) = 0 \tag{3.47}$$

where are initial conditions are

$$\dot{y}_2 = y_1, \quad y_2(0) = 1 \tag{3.48}$$

subject to the boundary condition $y(10) = -0.2$. In this form, there is no solution for arbitrary Ω, because the additional boundary condition over specifies the problem.

However, there is a discrete set of Ω for which a solution exists. The elements of this set are called eigenvalues of the differential equation.

The following code is an m-file called rhs.m which stores the differential equations:

```
rhs=rhs(t,y,dummy,omega)
rhs=[-omega*y(2); y(1)];
```

We create another m-file that solves the equations and plots the result:

```
y0=[1 0];                             %stores initial data
tspan=[0 10];                         %defines time domain
omega=-4;                             %fixes omega
[t,y]=ode45('rhs',tspan,y0,[], omega) %integrates equations
figure(1), plot(t,y(:,1))             %plots results
```

Matlab has a wide variety of ODE solvers. We use the standard `ode45` solver. Suppose we are interested in finding the solution with a boundary condition $y(10) = -0.2$. We may then vary Ω until our numerical solution matches the boundary condition. When the desired boundary condition is found, then we have solved for an eigenvalue of the boundary value problem. For example, if we set $\Omega = 3.055$, we find $y(10) = 0.2000$, hence Ω is an eigenvalue of the harmonic oscillator equation. We plot the solution in Figure (3.1) and turn attention to the more complicated equation (3.31).

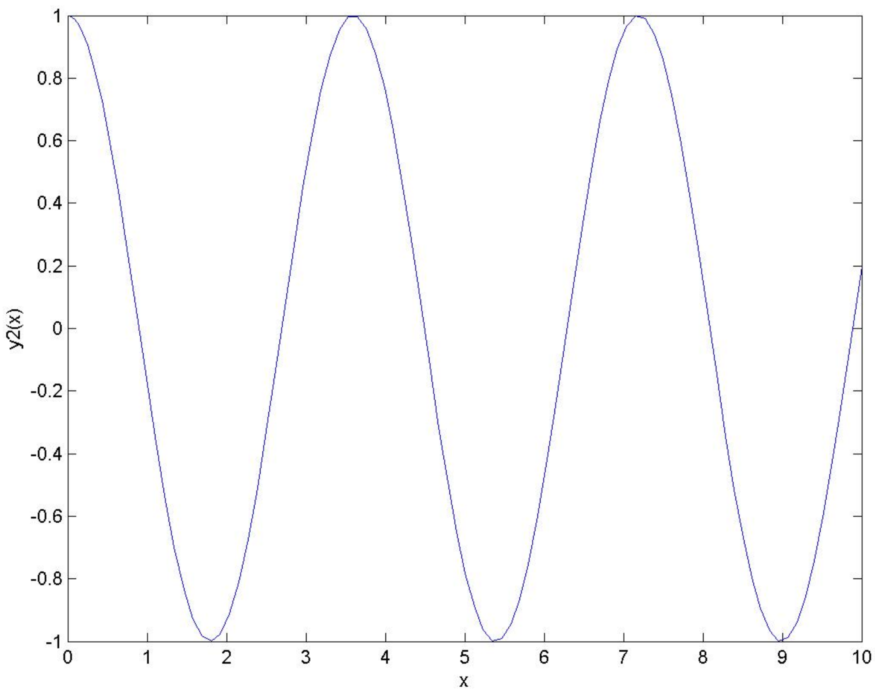

Figure 3.1 We plot the numerical solution $y_2(x)$ of (3.45) for $\Omega = 3.055$ and $x \in [0, 10]$. We infer that Ω is approximately a true eigenvalue of (3.45) since the numerical results satisfy our boundary condition.

3.4.2 Integration of the main Perturbation Equation

In analogy to our simple harmonic oscillator example, we decompose (3.31) into a system of first order equations

$$h_1' = \frac{1}{r}\left(1 + \frac{n}{f(r)} - \frac{4(n+2)k^2r^2}{2k^2r^2 + n(n+1)(r_+/r)^n}\right) h_1 \tag{3.49}$$

$$- \frac{1}{r^2}\left(\frac{k^2r^2}{f(r)}\frac{2k^2r^2 - n(n+3)(r_+/r)^n}{2k^2r^2 + n(n+1)(r_+/r)^n}\right) h_2 + \frac{\Omega^2}{f(r)^2} h_2 \tag{3.50}$$

$$h_2' = h_1 \tag{3.51}$$

subject to our near horizon boundary condition at $r_+ + \epsilon$

$$h_1(r_+ + \epsilon) = \frac{\Omega}{f} e^{\Omega r_*} \quad h_2(r_+ + \epsilon) = e^{\Omega r_*} \tag{3.52}$$

for $\epsilon = 10^{-6}$. We set $r_+ = 1$ using our rescaling freedom in our r coordinate. We integrate out from $r = 1 + \epsilon$ to $r = 200$ with an error tolerance of 10^{-6} and initial step of 10^{-6}. Note it is not possible to start the integration directly on the horizon due to the singular behavior of the perturbation equation. This is seen in the fact that $f(r) \to 0$ as $r \to r_+$. For fixed k, we find which values of Ω yield asymptotic sign change of h_z. Setting k fixes a mode in the z direction, and allows us to vary time modes. There are two independent solutions to (3.31); one decays exponentially as $r \to \infty$ and the other grows exponentially. This behavior is analogous to that exhibited by Bessel functions near the origin. The exponential behavior of our numerical method stems from the latter solution. We wish to preclude its contribution since h_z should be regular at spacial infinity.

By the intermediate value theorem, we know if there exist $\Omega_1 < \Omega_2$ where $h_z(r = 200; \Omega_1) < 0 < h_z^2(r = 200; \Omega_2)$, then there must exist an Ω such that $h_z(r = 200; \Omega) = 0$. This corresponds to a regular solution of the eigenvalue problem for all domain values. If $\Omega > 0$, then this corresponds to an unstable mode in the perturbation since the exponential factor of our perturbation metric will grow in time. For instance, note that for large r, h_z is decreasing in the green plot in Figure 3.2 where we numerically integrated the perturbation equation for $\Omega = 0.14$ and $k = 0.9$. When we perform the same integration for $\Omega = 0.15$ and $k = 0.9$, we find that $h_z(r)$ is increasing for large r (blue plot). Thus we infer that there is some $\Omega \in [0.14, 0.15]$, where h_z vanishes as r becomes large.

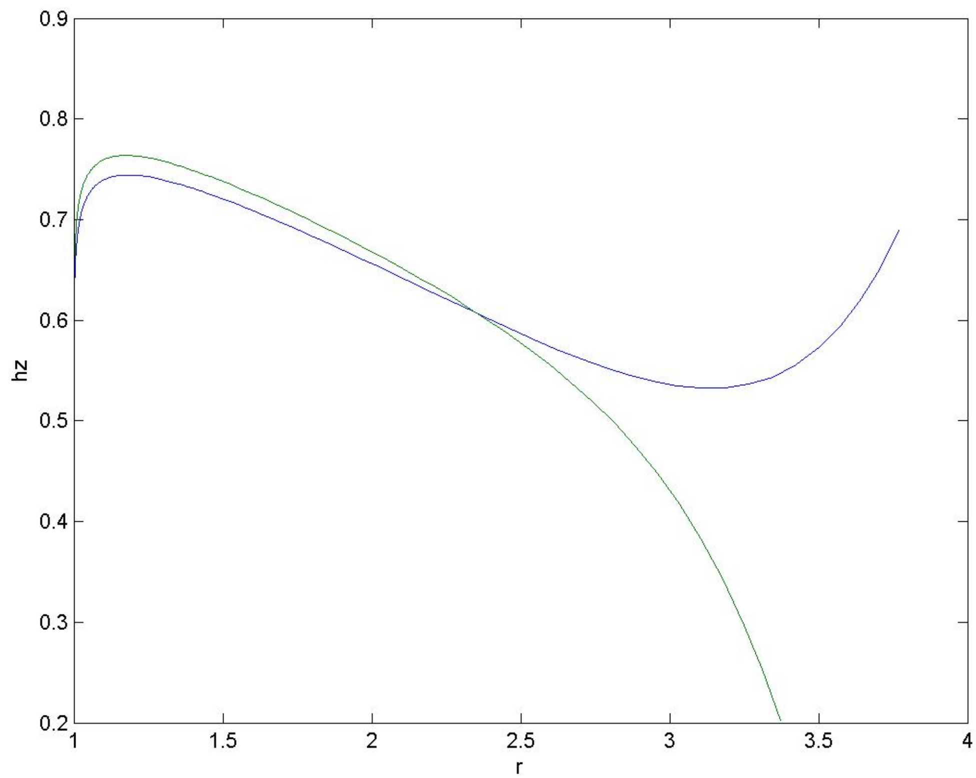

Figure 3.2 We plot the numerical solution $h_z(r)$ of the perturbation equation 3.31 for two parameter sets. We integrate from $r = r_+ = 1$ to $r = 200$ where $n = 1$ $k = 0.9$, $\Omega = 0.14$ (green) and $k = .9$, $\Omega = 0.15$ (blue). We plot our solution for $r \in [1, 4]$. Note how asymptotically the solution changes sign. This implies existence of a regular solution for some $\Omega \in [0.14, 0.15]$ since we know h_z is asymptomatically a modified Bessel function.

We preform this procedure for black strings in different dimensions and plot our results in Figure 3.3. Points in Figure 3.3 correspond to the values of k and Ω where asymptotic sign change of h_z occurs; hence they correspond to the unstable modes we are looking for.

Moreover, we find the largest k for which unstable modes exist for $D \in 5, \cdots, 9$ given by k_{max} according to the table

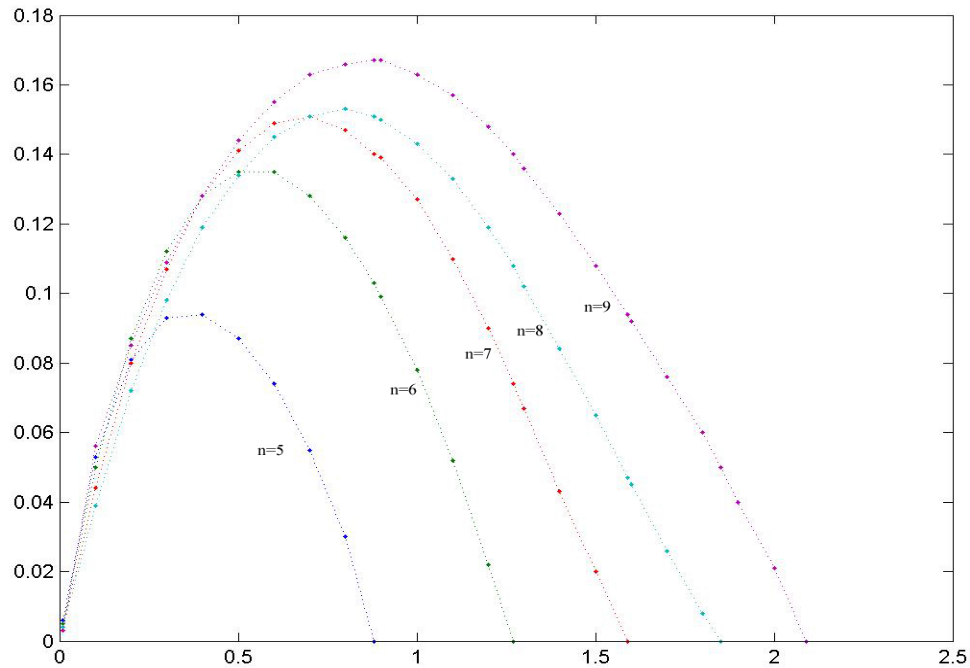

Figure 3.3 We plot pairs (k, Ω) that correspond to values of Ω where asymptotic sign change of numerical solutions of equation (3.31) occurs. These correspond to regular solutions of equation (3.31) and since $\Omega > 0$ imply unstable modes are found in black strings in $D = \{5, 6, 7, 8, 9\}$.

D	5	6	7	8	9
k_{max}	0.876159	1.268911	1.580761	1.848609	2.087136

Note there is a large range of k for which unstable Ω modes exist. For instance, if k is small (wavelengths are large) then we see unstable modes exists for all black strings. This result is analogous to large wavelength Jeans instabilities that were previously mentioned.

We are in exact agreement with [10] and [22]. We finally summarize our integration code. We input our system with

```
function rhs=rhs(r,h,dummy,omega,k)

rhs=[-1/r*(1+2/(1-1/r^2)-8*k^2*r^4/(k^2*r^4+3))*h(1)+1/r^2
```

```
*(k^2*r^2/(1-1/r^2)*(k^2*r^4-5)/(k^2*r^4+3))*h(2)+omega^2
/(1-1/r^2)^2*h(2); h(1)];
```

and provide the numerical numerical method with

```
del=10^(-6);
rst=1-del-atanh(1-del);
omega=.27;
k=.9;
h0=[omega/(1-1/(1+del)^2)*exp(omega*rst) exp(omega*rst) ];
rspan=[1+del 200];
options = odeset('RelTol',1e-6,'AbsTol',1e-6,'InitialStep',1e-6);
[r,h]=ode15s('myersshooteqns',rspan,h0, options,omega, k)
figure(1), plot(r,h(:,2))
```

We have confirmed that black strings in dimensions $D = \{5, 6, 7, 8, 9\}$ are not stable. It is natural to ask whether it may be possible to make such solutions stable in the presence of additional matter fields. We thus turn our attention to the charged analogue of black strings commonly called p-branes which are low energy solutions to string theory.

3.5 Charged p-branes

Charged p-branes may be heuristically thought of as extensions of the Reissner-Nordström solution to higher dimensions. There have been many works related to stability analyses of charged black p-branes. The first was the seminal work of Gregory-Laflamme [11] which concluded that all charged p-branes that are solutions to their low energy string model are linearly unstable except in the extremal limit [12]. Then the work of Reall in [26] and Hirayama, Kang, and Lee in [19] considered a more general string theory. In this generalized context, it was shown that linearly stable black p-branes can exist for sufficiently large non-extremal charge. We summarize the work of [12] and reproduce the work of [19] more accurately. Finally, we extend the work in [19] from $p = 4$ branes to the cases where $p = 1, 2, 3, 5, 6$.

3.5.1 Gregory Laflamme

Gregory and Laflamme were the first to consider the stability of charged black p-branes in [11]. These black p-branes are solutions to a ten dimensional theory with symmetry $\mathbb{R}^{10-D} \times \mathbb{R} \times S^{D-2}$ where $p = 10 - D$ denotes the dimension of the brane[2]. They are solutions of the low energy string theory given by the action

$$S_{GL} = \int d^{10}x\sqrt{-g}e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{2}{(D-2)!}F^2\right). \tag{3.53}$$

where F is a $(D-2)$ form analogous to the Maxwell tensor and our $D-2$ spherical symmetry is paired with the dimension of F. The action yields the equations of motion

$$\Box\phi - 2(\nabla\phi)^2 + F^2\frac{D-3}{(D-2)!} = 0 \tag{3.54}$$

$$\nabla_{a_1}(e^{-2\phi}F^{a_1 \ldots a_{D-2}}) = 0 \tag{3.55}$$

$$R_{ab} + 2\nabla_a\nabla_b\phi - \frac{2}{(D-3)!}F_{aa_2\cdots a_{D-2}}F_b^{a_2\cdots a_{D-2}} = 0. \tag{3.56}$$

The authors give the spherically symmetric solutions

$$ds^2 = -e^A dt^2 + Ee^G dr^2 + e^B dx_i dx^i + C^2 d\Omega^2_{D-2} \tag{3.57}$$

where

[2] Our D is not the same in this section as the previous section.

$$e^A = \frac{1-(r_+/r)^{D-3}}{1-(r_-/r)^{D-3}}, \quad e^{-G} = \left[1-\left(\frac{r_+}{r}\right)^{D-3}\right]\left[1-\left(\frac{r_-}{r}\right)^{D-3}\right], \quad B=0$$

$$C = r, \quad e^{-2\phi} = 1-\left(\frac{r_-}{r}\right)^{D-3} \tag{3.58}$$

and

$$F = Q\epsilon, \quad Q^2 = \frac{D-3}{2}(r_+r_-)^{D-3} \tag{3.59}$$

where r_+ and r_- are constants of integration corresponding to outer and inner event horizons, $3 < D < 10$, and ϵ is the volume form on S^{D-2}. Gregory and Laflamme show these charged black p-branes are linearly unstable by methods similar to those that were used to investigate the linear instability of black strings. In [12], the authors show that the instability vanishes in the case of extremal black p-branes. That is, p-branes with the maximal amount of charge before a naked singularity occurs.

3.5.2 Reall and Hirayama et al.

In [26] and [19], the respective authors investigate the stability of solutions to a more general low energy string theory than Gregory-Laflamme. In particular, it is shown in [19] that GL instabilities cease to exist for a wide class of charged p-branes. We use the notation of [19] and analyze the stability of black p-brane solutions to the theory given by the action

$$S_R = \int d^Dx\sqrt{-\bar{g}}\left[e^{-\bar{\beta}\bar{\phi}}\left(\bar{R}-\bar{\gamma}(\partial\bar{\phi})^2\right)-\frac{1}{2n!}e^{\bar{\alpha}\bar{\phi}}F_n^2\right] \tag{3.60}$$

where F_n^2 is given by

$$F_n^2 = F_{a_1\cdots a_n}F^{a_1\cdots a_n} \tag{3.61}$$

and F_n is an n-form field, $\bar{R}$ is the scalar curvature and $\bar{\phi}$ is a scalar dilaton field[3]. Moreover, we demand that F_n is exact (homologous to zero) and thus defines an $n-1$ form $A_{(n-1)}$ given by $F_n = dA_{(n-1)}$. $\bar{\beta}$, $\bar{\gamma}$, and $\bar{\alpha}$ are constants. Because all string theories are invariant under arbitrary conformal transformations we may define a conformal metric $\bar{g}_{MN} = e^{2\bar{\beta}\bar{\phi}/(D-2)}\hat{g}_{MN}$ and rescale the dilaton $\bar{\phi} = \phi/\sqrt{2\hat{\gamma}}$ which generates the equivalent action

$$S_R^c = \int d^D x \sqrt{-\hat{g}} \left[\hat{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2n!} e^{a\phi} F_n^2 \right] \tag{3.62}$$

where the parameter

$$a = \frac{\bar{\alpha} + (D-2n)\bar{\beta}/(D-2)}{\sqrt{2\hat{\gamma}}} \tag{3.63}$$

controls the coupling of the dilaton to F_n^2. The parameter a will play a central role in distinguishing under what conditions the GL instabilities vanish. We note that the GL action S_{GL} is given by the action (3.60) in the case where $\bar{\beta} = 2$, $\bar{\gamma} = -4$, $\bar{\alpha} = -2$, and $D = 10$.

Solutions to the equations of motion which come from equation (3.62) have been given in [7], [8], [14], and [21]. These solutions take the form

$$d\hat{s}^2 = -\left(1 + \frac{k}{r^{\tilde{d}}}\sinh^2\mu\right)^{-\frac{4\tilde{d}}{\triangle(D-2)}} + \left(1 + \frac{k}{r^{\tilde{d}}}\sinh^2\mu\right)^{\frac{4d}{\triangle(D-2)}} \left(\frac{dr^2}{U} + r^2 d\Omega_n^2\right) \tag{3.64}$$

$$+ \left(1 + \frac{k}{r^{\tilde{d}}}\sinh^2\mu\right)^{-\frac{4\tilde{d}}{\triangle(D-2)}} \delta_{ij} dz^i dz^j \tag{3.65}$$

$$e^{-\frac{\triangle}{2a}\phi} = 1 + \frac{k}{r^{\tilde{d}}}\sinh^2\mu, \quad U = 1 - \frac{k}{r^{\tilde{d}}}, \quad \triangle = a^2 + \frac{2d\tilde{d}}{D-2} \tag{3.66}$$

where $\tilde{d} = n-1$, $d = p+1$, and $D = \tilde{d} + d + 2 = 2 + n + p$ with coordinates $\{x^M\} = \{x^\nu, z^i\} = \{t, r, x^m, z^i\}$ where $m = 1, \cdots, n$ and $i = 1, \cdots, p$. The z^i

[3] Now D is the dimension of the full spacetime

coordinates denote extra spacial dimensions in the D dimensional spacetime and the x^m coordinates are the standard coordinates on the n-sphere.

A simplification comes by considering another conformal transformation given by $\hat{g}_{MN} = e^{2(n-1)\phi/(D-2)a} g_{MN}$. In [26] it is shown that this transformation will ultimately simplify the perturbation equations in the upcoming analysis. The transformed action becomes

$$S_R' = \int d^D x\sqrt{-g}\left[e^{-\beta\phi}\left(R - \gamma(\partial\phi)^2\right) - \frac{e^{\alpha\phi}}{2n!}F_n^2\right] \tag{3.67}$$

with constants of the form

$$\beta = \frac{1-n}{a}, \quad \gamma = \frac{1}{2} - \frac{(D-1)(n-1)^2}{(D-2)a^2}, \quad \alpha = a + \frac{(D-2n)(n-1)}{(D-2)a}. \tag{3.68}$$

The equations of motion are then given by

$$\nabla_M\left[e^{\alpha\phi}F^{MN_1\cdots N_{n-1}}\right] = 0 \tag{3.69}$$

$$\nabla^2\phi - \beta(\partial\phi)^2 = \frac{a}{2n!}e^{(\alpha+\beta)\phi}F^2 \tag{3.70}$$

$$\begin{aligned} R_{MN} &= (\gamma+\beta^2)\partial_M\partial_N\phi - \beta\nabla_M\nabla_N\phi \\ &+ \frac{1}{2(n-1)!}e^{(\alpha+\beta)\phi}F_{MP_1\cdots P_{n-1}}F_N^{P_1\cdots P_{n-1}} \end{aligned} \tag{3.71}$$

Applying the conformal transformation to our p-brane solutions, we find the transformed solutions take the form

$$ds^2 = -Udt^2 + V^{-1}dr^2 + R^2 d\Omega_n^2 + \delta_{ij}dz^i dz^j \tag{3.72}$$

where

$$V^{-1} = \frac{(1 + \frac{k}{r^{\tilde{d}}}\sinh^2\mu)^{\frac{4}{\Delta}}}{1 - k/r^{\tilde{d}}}, \quad R^2 = \left(1 + \frac{k}{r^{\tilde{d}}}\sinh^2\mu\right)^{\frac{4}{\Delta}} r^2 \tag{3.73}$$

and U is as previously defined. It is shown in [8] that the ADM mass M of the metric and charge density Q are given by

$$M = k\left(\tilde{d} + 1 + \frac{4\tilde{d}}{\triangle}\sinh^2\mu\right), \quad Q = \frac{\tilde{d}k}{\sqrt{\triangle}}\sinh 2\mu \tag{3.74}$$

with k and μ constants of integration.

The solutions in [11] are given by $a = (1-n)/2$. We note that the action (3.60) is invariant under the transformation $a \to -a$, $\phi \to -\phi$ so it will suffice to consider $|a|$ values in the upcoming numerical analysis.

We now proceed to analyze the stability of these solutions by perturbing the metric via $g_{MN} = \tilde{g}_{MN} + \epsilon h_{MN}$ where we assume

$$h_{MN} = e^{\Omega t + i m_i z^i} H_{MN}(r, x^m) \tag{3.75}$$

$$\delta\phi = e^{\Omega t + i m_i z^i} f(r, x^m) \tag{3.76}$$

$$\delta F = e^{\Omega t + i m_i z^i} \delta F(r, x^m) \tag{3.77}$$

and make an appropriate gauge transformation to set $H_{\mu i} = H_{ij} = 0$ where $i \neq j$. Moreover, we assume again that the perturbation is spherically symmetric which requires

$$H_{tm} = H_{rm} = 0 \quad H^m_n = K(r)\delta^m_n. \tag{3.78}$$

The equation of motion for the perturbed n-form field is given by

$$\nabla_N(e^{\alpha\phi}\delta F^{NP_1\cdots P_{n-1}}) = 0. \tag{3.79}$$

In [12], Gregory and Laflamme argue that we may set $\delta F = 0$ because there are no nontrivial solutions for the perturbed Maxwell field. We are thus left with the dilaton perturbation equation

$$\nabla^2 f - 2\beta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu f - H^{\mu\nu}\nabla_\mu\nabla_\nu\phi + \beta H^{\mu\nu}\nabla_\nu\phi\nabla_\nu\phi - \nabla_\mu\phi\nabla_\nu\left(H^{\mu\nu} - \frac{1}{2}H^\rho_\rho g^{\mu\nu}\right)$$

$$+ \frac{a}{2(n-1)!}e^{(\alpha+\beta)\phi}\left[H^{\mu\nu}F_{\mu\rho_1\ldots\rho_{n-1}}F_\nu^{\rho_1\ldots\rho_{n-1}} - \frac{\alpha+\beta}{n}F^2 f\right] = m^2 f \tag{3.80}$$

where Greek indices run over r and t, and the spacial Einstein equations are

$$\nabla^2 H_{\mu\nu} - 2\nabla_{(\mu}\nabla^\rho H_{\nu)\rho} + \nabla_\mu\nabla_\nu H^\rho_\rho - 2R_{\rho(\mu}H^\rho_{\nu)} + 2R_{\mu\rho\nu\sigma}H^{\rho\sigma}$$

$$+\beta\left(2\nabla_{(\mu}H^\rho_{\nu)} - \nabla^\rho H_{\nu\mu}\right)\nabla_\rho\phi - 2\beta\nabla_\mu\nabla_\nu f + 4(\gamma+\beta^2)\nabla_{(\mu}\phi\nabla_{\nu)}f$$

$$-\frac{1}{(n-1)!}e^{(\alpha+\beta)\phi}\left[(n-1)H^{\rho\sigma}F_{\mu\rho\lambda_1\ldots\lambda_{n-2}}F_{\nu\sigma}^{\lambda_1\ldots\lambda_{n-2}} - (\alpha+\beta)fF_{\mu\lambda_1\ldots\lambda_{n-1}}F_\nu^{\lambda_1\ldots\lambda_{n-1}}\right]$$

$$= m^2 H_{\mu\nu} \tag{3.81}$$

$$\nabla_\nu H^\nu_\mu - \beta H^\nu_\mu\nabla_\nu\phi - 2(\gamma+\beta^2)f\nabla_\nu\phi = 0 \tag{3.82}$$

$$H^\mu_\mu - 2\beta f = 0 \tag{3.83}$$

where Latin indices run over extra spacial dimensions. We note that all covariant derivatives are defined with respect to the Levi-Civita connection of the background metric. We only need to consider equations for H_{tt}, H_{tr}, H_{rr}, and K, since they determine the dilaton [26].

We recall that in our previous stability analysis of black strings, we varied Ω to find regular solutions of an equation for a perturbation component. We could perform an analogous analysis here where we would need to vary additional parameters related to our more general theory. In [26], Reall argues that the necessary and sufficient condition for a Gregory Laflamme instability to exist is that there should exist a static threshold unstable mode defined by $\Omega = 0$. We will thus take $\Omega = 0$ which has the consequence $H_{tr} = 0$. This simplifies the form of the perturbation to

$$H^M_N = \mathrm{diag}(\varphi(r), \psi(r), \chi(r), \cdots, \chi(r)) \tag{3.84}$$

and the perturbed metric for threshold modes may be written

$$ds^2 = -U(1+\varphi e^{-m_i z^i})dt^2 + V^{-1}(1+\psi e^{im_i z^i})dr^2 + R^2(1+\chi e^{im_i z^i})d\Omega_n^2 + dz^2. \tag{3.85}$$

The perturbation equations now become

$$\varphi'' + \left(\frac{U'}{2U} + \frac{V'}{2V} + n\frac{R'}{R} - \beta\phi'\right)\varphi' + \left[\frac{U''}{U} - \frac{U'}{U}\left(\frac{U'}{2U} - \frac{V'}{2V} - n\frac{R'}{R} + \beta\phi'\right) - \frac{m^2}{V}\right]\varphi$$
$$- \frac{U'}{U}\psi' - \left[\frac{U''}{U} - \frac{U'}{U}\left(\frac{U'}{2U} - \frac{V'}{2V - n\frac{R'}{R} + \beta\phi'}\right)\right]\psi = 0 \tag{3.86}$$

$$\psi'' + \left(\frac{U'}{2U} + \frac{V'}{2V} + n\frac{R'}{R} - \frac{2\gamma + 3\beta^2}{\beta}\phi'\right)\psi' + \frac{m^2}{V}\psi - \left(\frac{U'}{U} + 2\frac{\gamma+\beta^2}{\beta}\phi'\right)\varphi'$$
$$- 2n\left(\frac{R'}{R} + \frac{\gamma+\beta^2}{\beta}\phi'\right)\chi' = 0 \tag{3.87}$$

$$\psi' + \left(\frac{U'}{2U} + n\frac{R'}{R} - \frac{\gamma+2\beta^2}{\beta}\phi'\right)\psi - \left(\frac{U'}{2U} + \frac{\gamma+\beta^2}{\beta}\phi'\right)\varphi - n\left(\frac{R'}{R} + \frac{\gamma+\beta^2}{\beta}\phi'\right)\chi = 0. \tag{3.88}$$

We can eliminate χ to find two second order coupled equations for φ and ψ given by

$$r(r^{\widetilde{d}}-k)\varphi''+\left[(\widetilde{d}+1)r^{\widetilde{d}}-k\right]\varphi'-m^2r^{\widetilde{d}+1-4\widetilde{d}/\triangle}(r^{\widetilde{d}}+k\sinh^2\mu)^{4/\triangle}\varphi-\widetilde{d}k\psi'=0 \quad (3.89)$$

$$\begin{aligned}
&r^2(r^{\widetilde{d}}-k)^2\left(r^{\widetilde{d}}+k\sinh^2\mu\right)\psi''+r(r^{\widetilde{d}}-k)^2\Bigg[2\widetilde{d}\left(r^{\widetilde{d}}-\frac{2}{\triangle}k\sinh^2\mu\right)\\
&-(\widetilde{d}-3)\left(r^{\widetilde{d}}+k\sinh^2\mu\right)\Bigg]\psi'-\Bigg\{m^2r^{\widetilde{d}+2-4\widetilde{d}/\triangle}(r^{\widetilde{d}}-\frac{2}{\triangle}k\sinh^2\mu)^{1+4/\triangle}\\
&+\widetilde{d}k\left[W+\frac{2}{\triangle}\left(2\widetilde{d}^2+(\widetilde{d}+3)(a^2-\frac{2\widetilde{d}^2}{D-2})\right)\sinh^2\mu(r^{\widetilde{d}}-k)^2\right]\Bigg\}\psi+\widetilde{d}kW\varphi \quad (3.90)
\end{aligned}$$

where we have set

$$W=\widetilde{d}(r^{\widetilde{d}}-k)(r^{\widetilde{d}}+k\sinh^2\mu)-\frac{2}{\triangle}\left(a^2-\frac{2\widetilde{d}^2}{D-2}\right)\sinh^2\mu(r^{\widetilde{d}}-k)^2+\widetilde{d}k\cosh^2\mu r^{\widetilde{d}} \quad (3.91)$$

and defined

$$m^2=\sum_i m_i^2. \quad (3.92)$$

Now the black p-brane stability question is equivalent to finding an m for which the above equations admit non singular solutions outside the event horizon. We need to determine the asymptotic and near horizon behavior of (3.89) and (3.90) in order to set up a numerical method. As pointed out in [19], we note at radial infinity φ and ψ have asymptotic behavior

$$\varphi(r)\approx e^{\pm mr}u_\pm(r)\approx e^{\pm mr}\left[r^{-\frac{\widetilde{d}+1}{2}}\mp\frac{(\widetilde{d}-1)(\widetilde{d}+1)}{8m}r^{-\frac{\widetilde{d}+3}{2}}+\cdots\right] \quad (3.93)$$

$$\psi(r) \approx e^{\pm mr} v_{\pm}(r) \approx e^{\pm mr} \left[r^{-\frac{\tilde{d}+3}{2}} \mp \frac{(\widetilde{d}+1)(\widetilde{d}+3)}{8m} r^{-\frac{\tilde{d}+5}{2}} + \cdots \right] \tag{3.94}$$

We define a near horizon parameter $\rho = 1 + \epsilon$ and a scaled mode $\bar{m} = r_+ m$. We list the four possible near horizon behaviors of φ and ψ along with their asymptotic values

$$\psi_I \approx \epsilon + O(\epsilon^2), \quad A_1 e^{-\overline{m}\rho} v_-(\rho) + B_I e^{\overline{m}\rho} v_+(\rho) \tag{3.95}$$

$$\psi_{II} \approx 1 + O(\epsilon^2), \quad A_{II} e^{-\overline{m}\rho} v_-(\rho) + B_{II} e^{\overline{m}\rho} v_+(\rho) \tag{3.96}$$

$$\psi_{III} \approx \epsilon \ln \epsilon + O(\epsilon^2 \ln \epsilon), \quad A_{III} e^{-\overline{m}\rho} v_-(\rho) + B_{III} e^{\overline{m}\rho} v_+(\rho) \tag{3.97}$$

$$\psi_{IV} \approx -\epsilon^{-1} - \frac{\overline{m}^2}{\widetilde{d}} (\cosh\mu)^{8/\triangle} \ln \epsilon + O(\epsilon^2), \quad A_{IV} e^{-\overline{m}\rho} v_-(\rho) + B_{IV} e^{\overline{m}\rho} v_+(\rho) \tag{3.98}$$

$$\varphi_I \approx \epsilon + O(\epsilon^2), \quad \overline{A}_1 e^{-\overline{m}\rho} u_-(\rho) + \overline{B}_I e^{\overline{m}\rho} u_+(\rho) \tag{3.99}$$

$$\varphi_{II} \approx 1 + O(\epsilon^2), \quad \overline{A}_{II} e^{-\overline{m}\rho} u_-(\rho) + \overline{B}_{II} e^{\overline{m}\rho} u_+(\rho) \tag{3.100}$$

$$\varphi_{III} \approx \epsilon \ln \epsilon + O(\epsilon^2 \ln \epsilon), \quad \overline{A}_{III} e^{-\overline{m}\rho} u_-(\rho) + \overline{B}_{III} e^{\overline{m}\rho} u_+(\rho) \tag{3.101}$$

$$\varphi_{IV} \approx -\epsilon^{-1} - \frac{\overline{m}^2}{\widetilde{d}} (\cosh\mu)^{8/\triangle} \ln \epsilon + O(\epsilon^2), \quad \overline{A}_{IV} e^{-\overline{m}\rho} u_-(\rho) + \overline{B}_{IV} e^{\overline{m}\rho} u_+(\rho). \tag{3.102}$$

We require that ψ and φ to decrease exponentially as $r \to \infty$ since they are small perturbations of our fixed metric. According to [26], we need φ and ψ to be regular at the horizon, which precludes IV from being viable boundary conditions. We note that III produce a logarithmic curvature singularity at the horizon whereas I and II do not. Thus requiring regularity at the horizon is equivalent to demanding φ and ψ have near horizon behavior of a linear combinations of I and II. We now begin our search for regular solutions of (3.89) and (3.90) that satisfy these boundary conditions. As in [19], we will start a numerical integration near the horizon where

$$\psi = C\psi_I + E\psi_{II}, \quad \varphi = C\varphi_I + E\varphi_{II} \tag{3.103}$$

which has asymptotic behavior

$$\psi \approx (CA_I + EA_{II})e^{-\bar{m}\rho}v_-(\rho) + (CB_I + EB_{II})e^{\bar{m}\rho}v_+(\rho) \tag{3.104}$$

$$\varphi \approx (C\bar{A}_I + E\bar{A}_{II})e^{-\bar{m}\rho}u_-(\rho) + (C\bar{B}_I + E\bar{B}_{II})e^{\bar{m}\rho}u_+(\rho) \tag{3.105}$$

We seek a combination of C, E, and $\bar{m}$ such that the coefficients of the positive exponential terms vanish, which requires

$$CB_I + EB_{II} = 0, \quad C\bar{B}_I + E\bar{B}_{II} = 0. \tag{3.106}$$

This could be done by varying C and E and searching for such a solution via numerical integration, or equivalently checking that

$$P(\bar{m}, \mu, a, \tilde{d}, D) \equiv B_I\bar{B}_{II} - B_{II}\bar{B}_I \tag{3.107}$$

vanishes for some value of $\bar{m}$ for fixed μ, a, $\tilde{d}$, and D. We use the latter procedure in the subsequent section.

3.6 p-brane Numerics

We perform our numerical analysis of the perturbation equations in Mathematica 6.0 but first consider a test problem. We consider a coupled system of second order equations given by

$$\ddot{x}(t) = -y(t), \quad \ddot{y}(t) = -2x(t)^2 \tag{3.108}$$

$$x(0.01) = y(0.01) = \dot{x}(0.01) = \dot{y}(0.01) = 1 \tag{3.109}$$

and we use the Mathematica NDsolve command

```
In[1]:= s = NDSolve[{x''[t] == -y[t], y''[t] == -2 x[t]^2,
x[.01] == y[.01] == 1, x'[.01] == 1, y'[.01] == 1}, {x, y}, {t,1}];
Plot[Evaluate[{x[t], y[t]} /. s], {t, .01, 1}]
```

to numerically integrated the coupled system and plot the results in Figure (3.4).

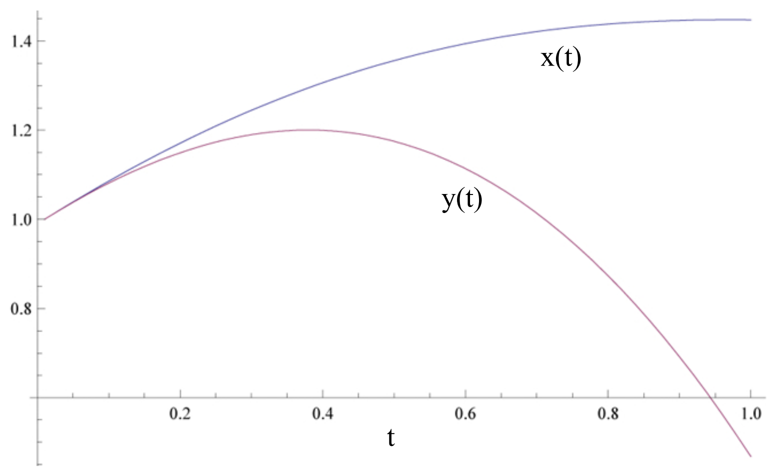

Figure 3.4 We plot numerical solutions to the test problem (3.108) subject to the boundary conditions (3.109) where $x(t)$ is the blue plot and $y(t)$ the purple plot.

Extending this example to the numerical integration of (3.89) and (3.90) is only slightly more complicated. We summarize our code for the integration of (3.89) and (3.90) in Appendix A. We perform stability analyses in $D = 10$ dimensions for six p-branes, $p = \{1, 2, 3, 4, 5, 6\}$. We fix a value of the parameter a which couples the dilaton ϕ to the n-form field F_n^2, and typically choose $a = \{0, 1/2, 1, 3/2, 2, 3\}$ for a given p. We are then left with two free parameters μ and m. We vary $\mu \in [0, 5]$ in increments of 0.01 and for fixed μ, integrate (3.89) and (3.90) for varying m in increments of 0.001 starting at $m = 0$, and check the sign of P in equation (3.107). When we find the first value of m where $P < 0$, we set $m = m^*$ and store the pair (m^*, μ), increment μ and repeat our method. We call m^* the threshold mass or threshold mode since it is the point where P changes sign and hence regular solutions to (3.89) and (3.90) exist.

3.7 Conclusions

We plot our numerical results in Figures (3.5) – (3.10) for $p = 1, \cdots, 6$ branes. Recall that the existence of a threshold mode (mass) is equivalent to the existence of a GL instability. The colored regions indicate areas where $P > 0$. The GL analysis corresponds to the plots that decay exponentially as μ becomes large and are given by $a_{GL} = (7 - p)/2$. We thus see that instabilities are always present in these instances up to their extremal limit of charge where it appears they vanish as claimed in [11] and [12].

We see that the coupling parameter a determines when a sufficient amount of charge will stabilize a p-brane. We use $a_{GL} = (7 - p)/2$ to classify stability behavior. For $a < a_{GL}$, all p-branes have a μ for which $P < 0$ for all m and hence the p-branes have no Gregory Laflamme instabilities. For $a > a_{GL}$, p-branes exhibit Gregory Laflamme instabilities for all μ.

Thus we have shown there is a wide class of uncharged p-branes depending on a coupling parameter $a < a_{GL}$ which exhibit linear instabilities. Moreover, these instabilities disappear in the presence of sufficiently large values of charge.

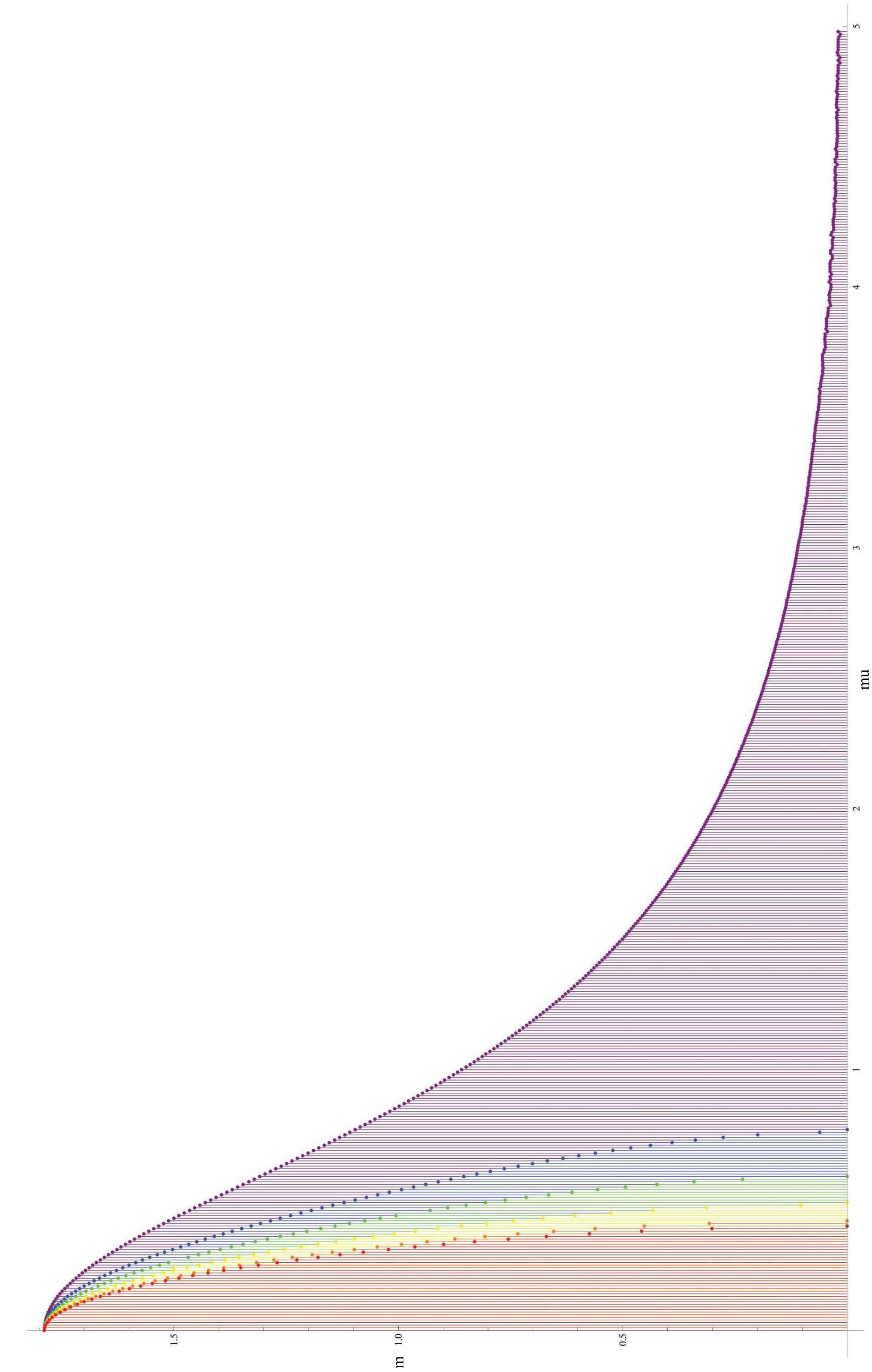

Figure 3.5 This is a plot of threshold masses for fixed μ of the φ, ψ perturbation equations which imply GL instabilities. We consider a $p = 1$ brane in $D = 10$ with ADM mass $M = 32$ for $a = \{0, 1/2, 1, 3/2, 2, 3\}$ increasing from red to purple with $\mu \in [0, 5]$ $m \in [0, 1.8]$. Note for small a and large charge parameter μ that GL instabilities vanish.

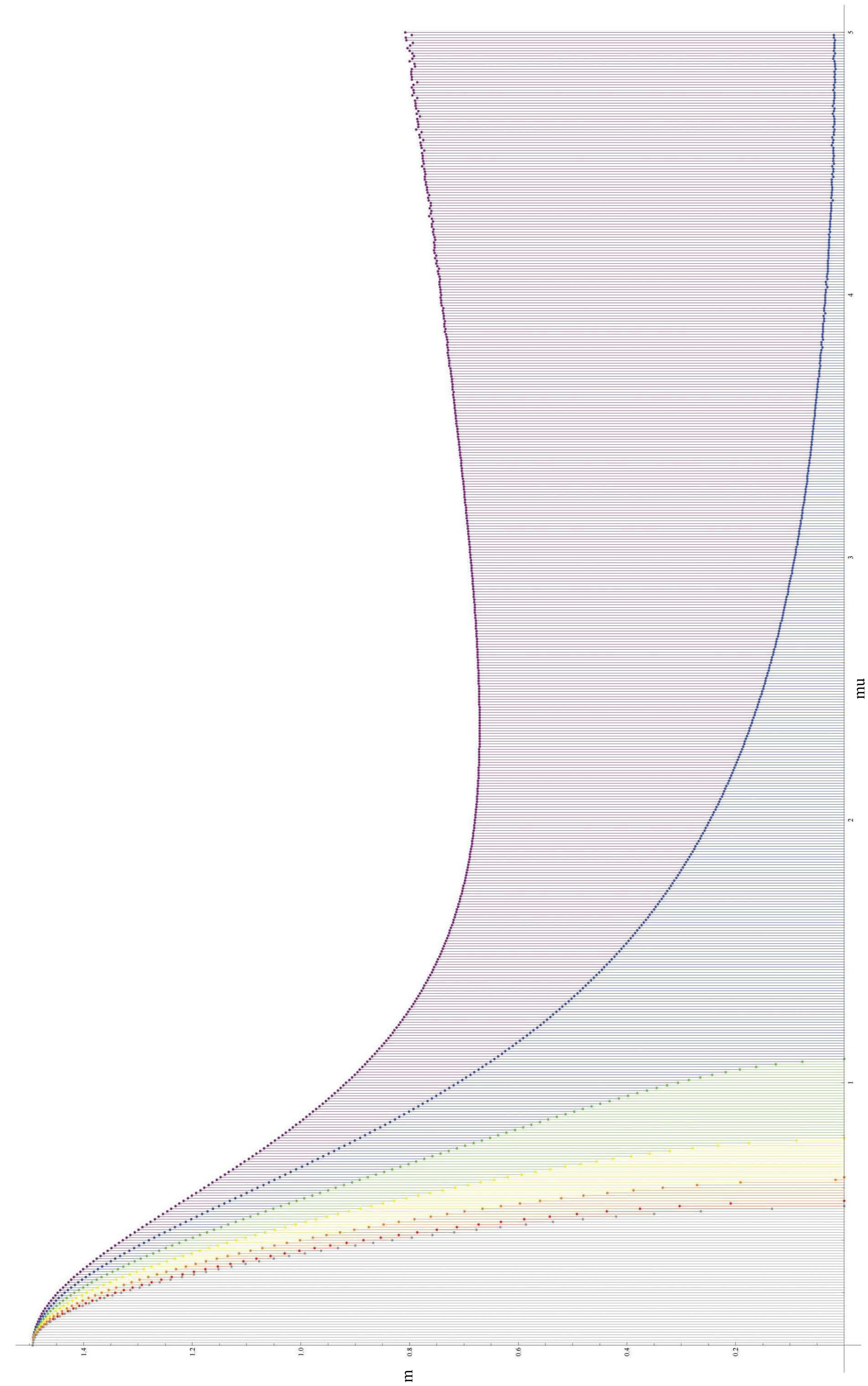

Figure 3.6 This is a plot of threshold masses for fixed μ of the φ, ψ perturbation equations which imply GL instabilities. We consider a $p = 2$ brane in $D = 10$ with ADM mass $M = 32$ for $a = \{0, 1/2, 1, 3/2, 2, 5/2, 3\}$ increasing from gray to purple with $\mu \in [0, 5]$ $m \in [0, 1.5]$. Note for small a and large charge parameter μ that GL instabilities vanish.

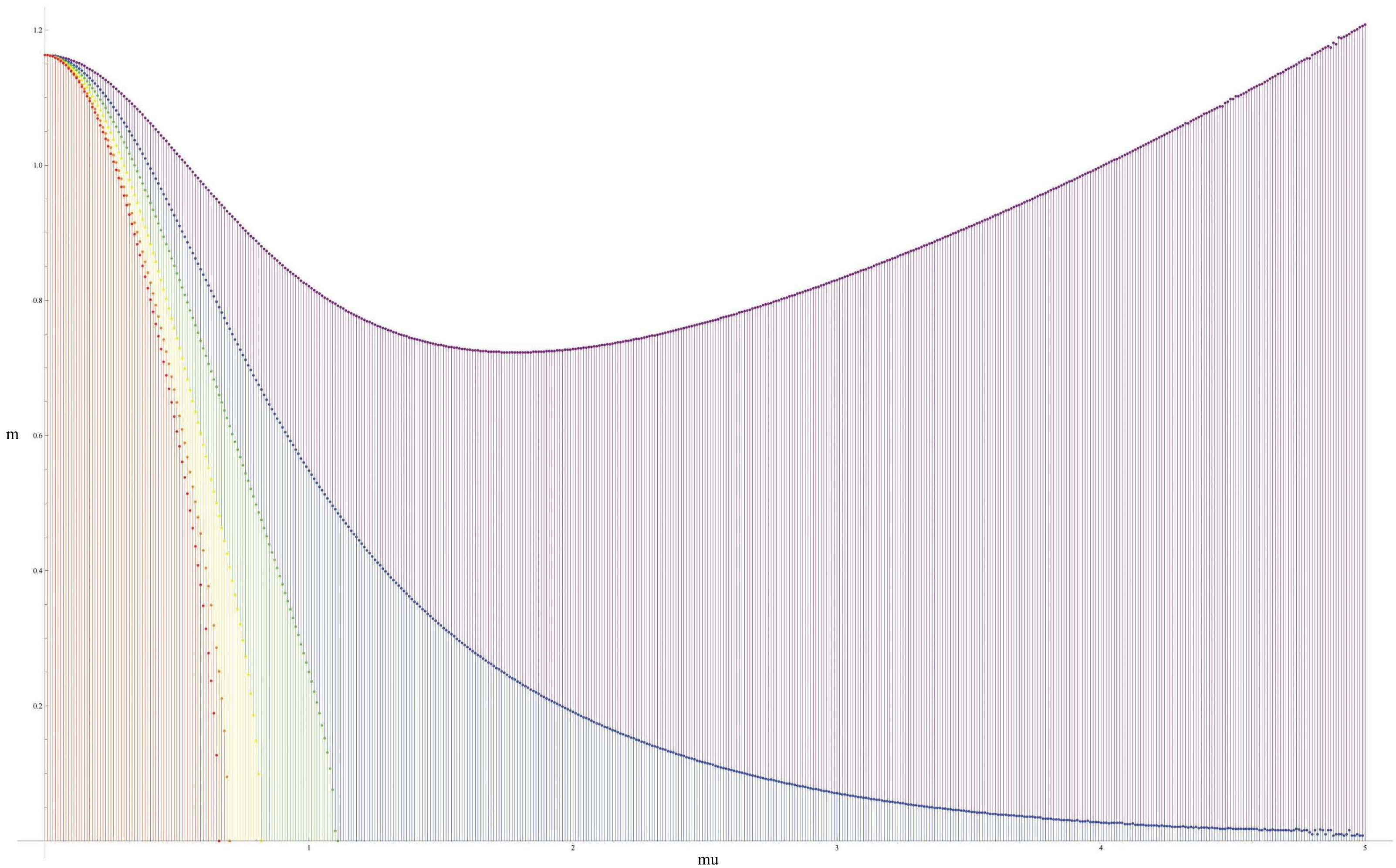

Figure 3.7 This is a plot of threshold masses for fixed μ of the φ, ψ perturbation equations which imply GL instabilities. We consider a $p = 3$ brane in $D = 10$ with ADM mass $M = 32$ for $a = \{0, 1/2, 1, 3/2, 2, 3\}$ increasing from red to purple with $\mu \in [0, 5]$ $m \in [0, 1.2]$. Note for small a and large charge parameter μ that GL instabilities vanish.

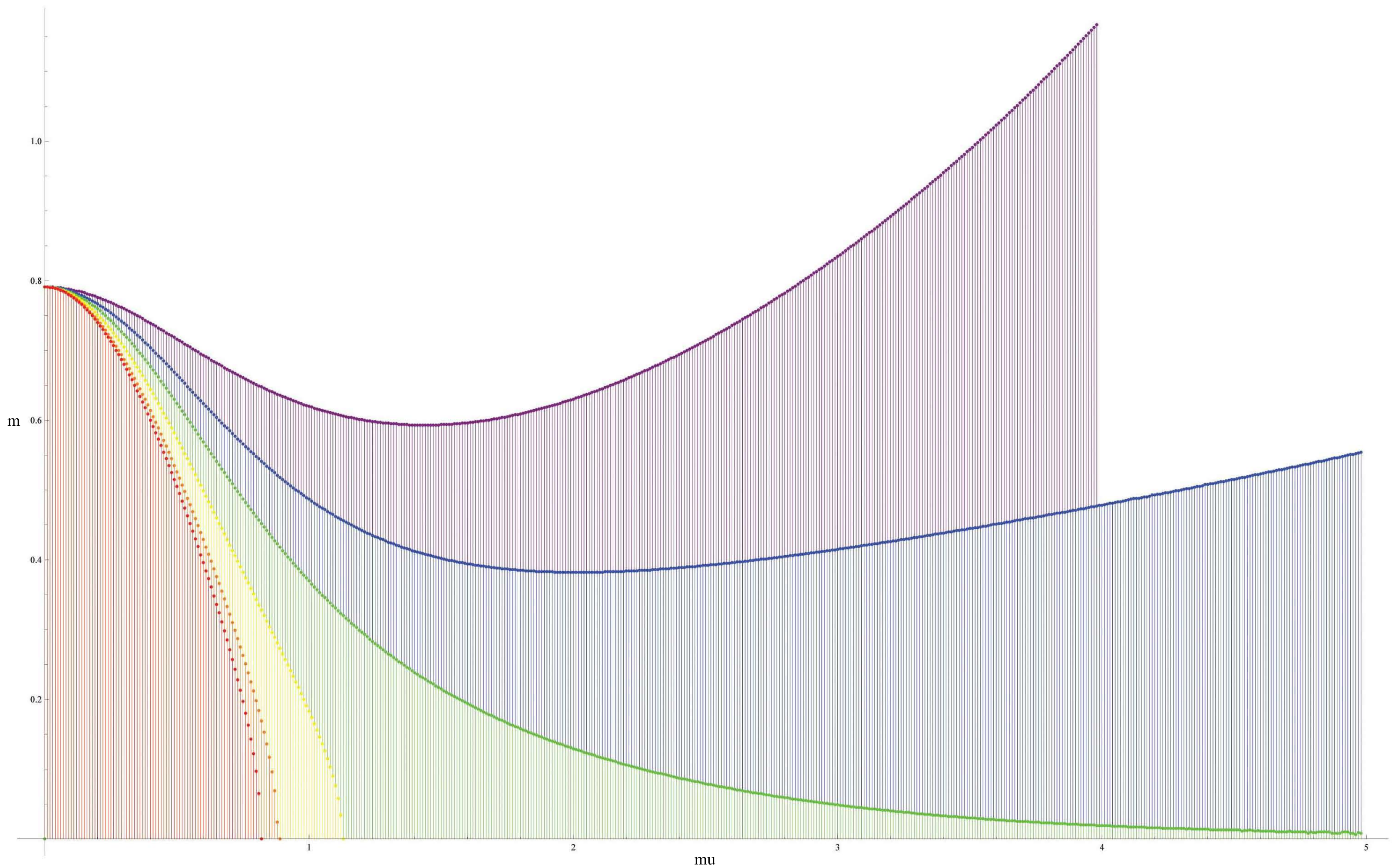

Figure 3.8 This is a plot of threshold masses for fixed μ of the φ, ψ perturbation equations which imply GL instabilities. We consider a $p = 4$ brane in $D = 10$ with ADM mass $M = 32$ for $a = \{0, 1/2, 1, 3/2, 2, 3\}$ increasing from red to purple with $\mu \in [0, 5]$ $m \in [0, 1.2]$. Note for small a and large charge parameter μ that GL instabilities vanish.

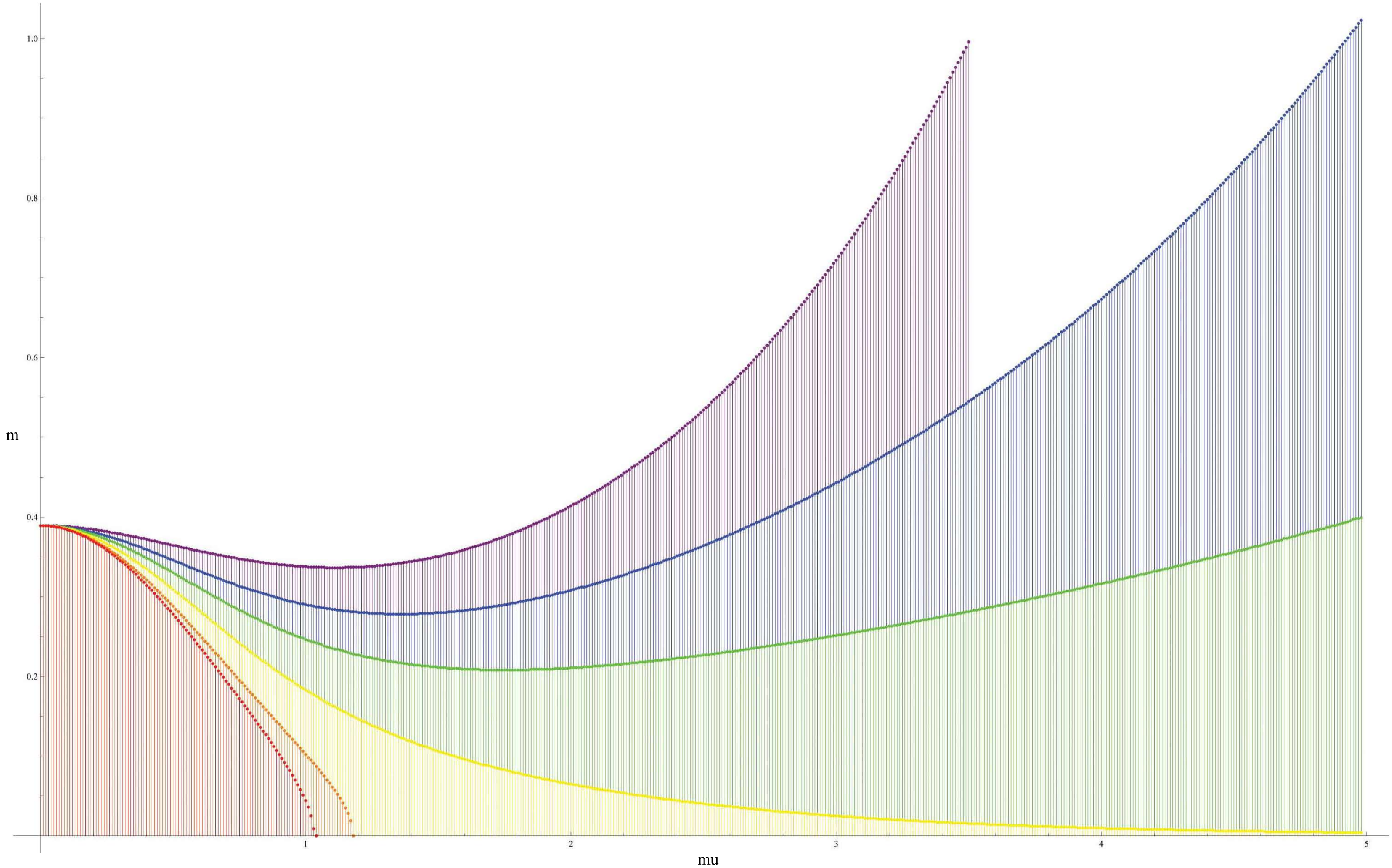

Figure 3.9 This is a plot of threshold masses for fixed μ of the φ, ψ perturbation equations which imply GL instabilities. We consider a $p = 5$ brane in $D = 10$ with threshold mass $M = 32$ for $a = \{0, 1/2, 1, 3/2, 2, 3\}$ increasing from red to purple with $\mu \in [0, 5]$ $m \in [0, 1.0]$ Note for small a and large charge parameter μ that GL instabilities vanish.

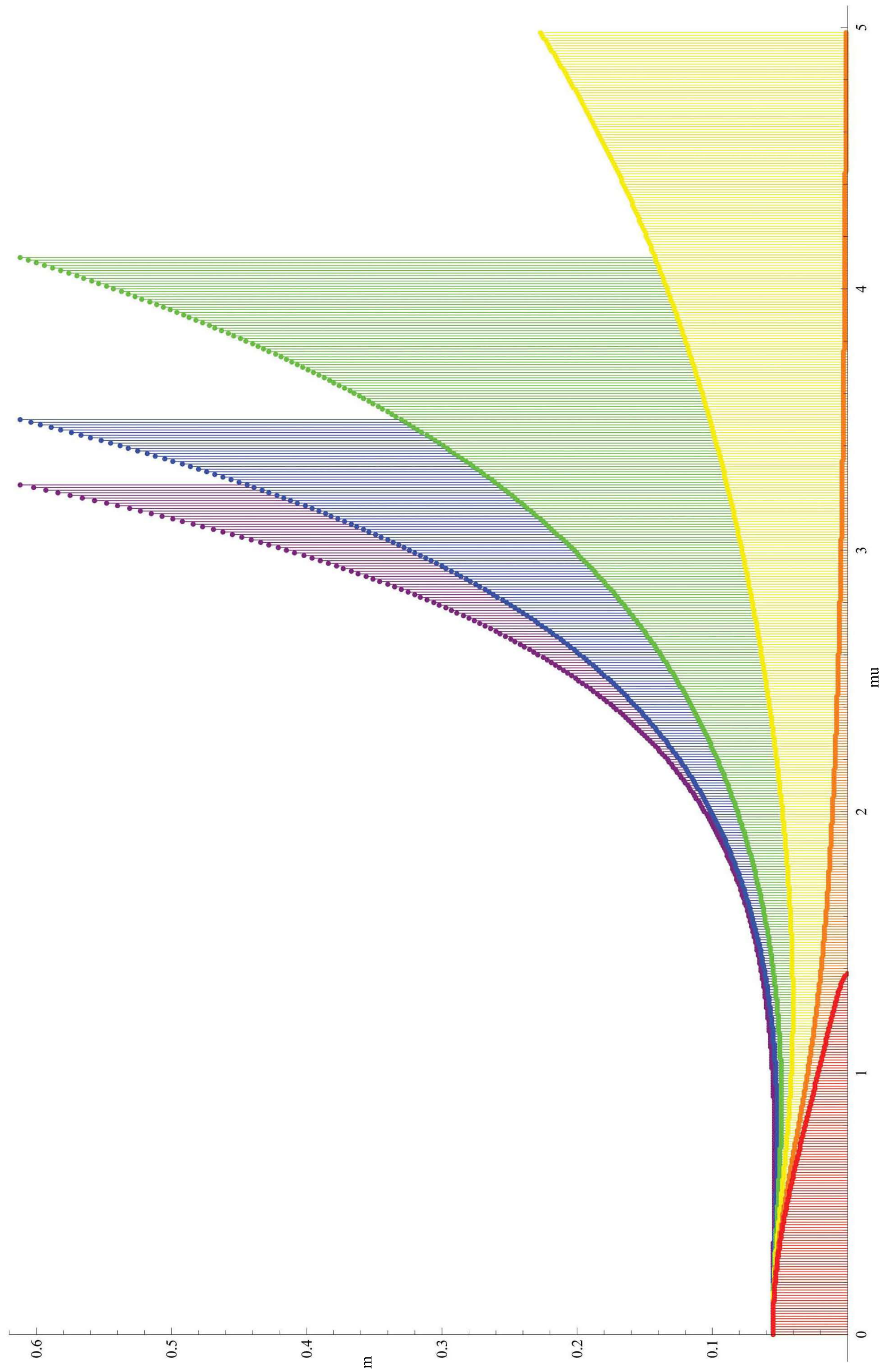

Figure 3.10 This is a plot of threshold masses for fixed μ of the φ, ψ perturbation equations which imply GL instabilities. We consider a $p = 6$ brane in $D = 10$ with ADM mass $M = 32$ for $a = \{0, 1/2, 1, 3/2, 2, 3\}$ increasing from red to purple with $\mu \in [0, 5]$ $m \in [0, 0.6]$. Note for small a and large charge parameter μ that GL instabilities vanish.

Chapter 4

The GRMHD Eigenvalue Problem

Up to now we have considered linear stability problems in higher dimensional relativity. We turn attention to the somewhat different problem of evolving compact objects with magnetic fields in general relativity. While not necessarily done in higher dimensions, the problem is similar to the linear stability analyses we previously presented. Specifically, one would like to know what happens to the matter fields around a black hole as it approaches its endstate (believed to be a Kerr spacetime). What one observes near a black hole is the time development of matter fields that either leave or fall into the black hole. From a numerical standpoint, it is of interest to model the radiation of these systems. Thus one wants to consider GRMHD. This chapter addresses questions related to this goal. Specifically, we consider the characteristic structure of the matter part of the GRMHD Einstein equations. We follow a technique introduced by Brio and Wu [4] for the Newtonian version of this problem.

Brio and Wu analyze the eigenvalue problem for the equations of Newtonian nonrelativistic ideal magnetohydrodynamics (MHD). This consists of calculating eigenvalues and eigenvectors of the MHD equations and examining cases where two eigenvalues or eigenvectors degenerate to a single one. These quantities give local information about a nonlinear theory. It is in a sense the easiest nontrivial information one can produce for the MHD equations.

The eigenvalues and eigenvectors of the Jacobian of a system are commonly used in numerical methods that require a spectral decomposition. This information is used to identify incoming and outgoing modes near the boundary of a numerical grid. Specifically, the spectral information can be used to reduce spurious reflections from the boundaries of a computational grid.

GRMHD represents general relativity (Einstein's equations) coupled to MHD. This framework is needed to model highly compact magnetically charged astrophysical objects such as magnetized neutron stars. We wish to extend the MHD analysis to GRMHD for purposes of numerical simulation. The full spectral decomposition of GRMHD equations

4.1 Introduction

The characteristic structure (eigenvalues and eigenvectors) of a system of partial differential equations determines the hyperbolicity of the system. This information also allows one to predict relevant wave speeds. This aides to improve the overall accuracy of numerical simulations of the equations. One place where this can be implemented is fixing problems near the boundaries of the grid. We seek to calculate the eigenvectors of the GRMHD equations and apply the normalization procedure in [4] to resolve this problem. We summarize information about hyperbolic evolution equations and the numerical and analytic methods of [4]. We then extend this analysis to a version of the GRMHD equations.

4.2 Hyperbolicity of Evolution Equations

Consider a system of evolution equations in one spacial variable given by

$$\partial_t u_i + \partial_x F_i = q_i, \quad i = 1, \cdots, n \tag{4.1}$$

where $F_i = F_i(u)$ and $q_i(u)$ are sufficiently differentiable functions and do not depend on derivatives of u. Equation (4.1) is a "balance law" since we have completely separated out first order time derivatives from spacial derivatives. In the event that the sources q_i vanish, then the equation is in conservative form; it takes the form of a continuity equation. We can write (4.1) more conveniently as

$$\partial_t u_i + \sum_j A_i^j \partial_x u_j = q_i, \quad i = 1, \cdots, n \tag{4.2}$$

where $A_i^j = \partial F_i / \partial u_j$ is the Jacobian matrix of the system. We will see that we can write the ideal MHD equations in the form (4.2).

Important aspects of the overall system can be learned from the properties of the matrix A. More specifically, let us study the characteristic structure of A_j^i. Let λ_i be the eigenvalues of A. We characterize the system of evolution equations as hyperbolic if λ_i are all real valued functions. We will further call the system strongly hyperbolic if there exists a complete set of eigenvectors associated with the λ_i. If all eigenvalues are real, but there does not exist a complete set of eigenvectors, we call the system weakly hyperbolic.

The reason strong hyperbolicity is significant can be seen as follows. Suppose we have a strongly hyperbolic system. We may then define functions w_i by

$$\mathbf{u} = R\mathbf{w} \tag{4.3}$$

where R is the matrix whose columns are the right eigenvectors of λ_i. Moreover, we have the diagonalization

$$RAR^{-1} = \text{diag}(\lambda_i), \tag{4.4}$$

provided R does not vanish. On diagonalization, the evolution equations for w_i are given by the simpler system of advection equations

$$\partial_t w_i + \lambda_i \partial_x w_i = q_i' \tag{4.5}$$

where the $q_i' = q_i'(w)$ are not functions of derivatives of w. We have thus reduced solving any strongly hyperbolic system to solving a system of advection equations. Strongly hyperbolic systems also have the nice property that they are well posed. That is, their solution depends continuously on initial data (no bifurcation) and solutions are locally unique.

The MHD equations form a weakly hyperbolic system. We will use a normalization process on the eigenvectors of MHD to force it into a strongly hyperbolic system. This procedure is necessary for the numerical method used to solve the equations.

4.3 Numerical Considerations

Our interest in the hyperbolicity in the MHD equations whether Newtonian or relativistic stems in part from a need to construct robust numerical algorithms that handle discontinuities (shocks) that arise in fluid simulations. Considering for a moment the Newtonain case as a simplification of the full relativistic equations, we summarize the work of Brio and Wu [4]. There work is mainly concerned with developing a numerical scheme called a Roe upwind differencing solver. We briefly summarize this numerical method because its use of symmetric hyperbolic systems (strong hyperbolic system with distinct eigenvalues) motivates our consideration of the GRMHD eigenvalue problems.

To begin, we define discetized time and space variables $x_i = i\Delta x$ and $t_n = n\Delta t$ and let v_i^n denote the approximate numerical solution to the system of equations

$$U_t + [F(U)]_x = 0 \tag{4.6}$$

for some vector of quantities U (for example the conservative variables we use later). We take a finite difference approximation to (4.6) given by

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{f_{i+1/2}^n - f_{i-1/2}^n}{\Delta x} = 0. \tag{4.7}$$

In the construction of the Roe scheme, we approximate (4.6) in each cell $(x_i, x_{i+1}) \times (t_n, t_{n+1})$ by

$$U_t + [G(U)]_x = 0 \tag{4.8}$$

where we set

$$[G(U)]_i = F_i + A_{i+1/2}(U_i - v_i^n), \quad F_i = F(U_i). \tag{4.9}$$

The matrix $A_{i+1/2}$ is called the Roe matrix and it the Jacobian of our system. It is given by the following conditions:

$$F_{i+1} - F_i = A_{i+1/2}(U_{i+1} - U_i) \quad \text{for all } U_i \text{ and } U_{i+1} \tag{4.10}$$

where it is assumed $A_{i+1/2}$ has real eigenvalues and a complete set of right eigenvectors and is given by

$$A_{i+1/2}(U_{i+1}, U_i) \to A(U_0) = \left.\frac{\partial F}{\partial U}\right|_{U=U_0} \quad \text{as } U_{i+1} \quad \text{and} \quad U_i \to U_0. \tag{4.11}$$

After determining $A_{i+1/2}$ by these conditions, we compute its eigenvalues $\lambda_k^{i+1/2}$ and right eigenvectors $R_k^{i+1/2}$. We then define decomposition coefficients $C_k^{i+1/2}$ by

$$v_{i+1} - v_i = \sum_k C_k^{i+1/2} R_k^{i+1/2} \tag{4.12}$$

and finally compute $f_{i+1/2}$ according to

$$f_{i+1/2} = \frac{1}{2}(F_i + F_{i+1}) - \frac{1}{2}\sum_k |\lambda_k^{i+1/2}| C_k^{i+/1/2} R_k^{i+1/2} \tag{4.13}$$

which gives v_i in (4.7). We note again that A must have a complete set of eigenvectors for this process to work. We thus proceed to calculate the eigenvalues and eigenvectors of the Jacobian matrix A of the MHD equations to check this condition.

4.4 The Newtonian Ideal MHD Eigenvalue Problem

The MHD equations model the flow of a conducting fluid **u** which interacts with a magnetic field **B**, and may be viewed as Maxwell's equations coupled to the equations of fluid dynamics in the limit on infinite conductivity (no electric field in the frame of the fluid). Neglecting displacement current, electrostatic forces, viscosity, resistivity, and heat conduction, the ideal magnetohydrodynamic equations are given by

$$\rho_t + \partial_i(\rho u^i) = 0 \tag{4.14}$$

$$(\rho u_i)_t + \partial_i(\rho u_i u_j + \delta_{ij} P^* - B_i B_j) = 0 \tag{4.15}$$

$$\partial_t B_j + \partial_i(u_i B_j - u_j B_i) = 0 \tag{4.16}$$

$$\partial_t E_i + \partial_i((E + P^*)u_i - B_i(B_j u^j)) = 0 \tag{4.17}$$

where we also have the no monopole constraint $\nabla \cdot \mathbf{B} = 0$ from the Maxwell equations. In the above ideal MHD equations ρ is the fluid density, P the static pressure, $P^* = P + \frac{1}{2}B_i B^i$ the full static pressure plus magnetic, $E = \frac{\rho}{2}u_i u^i + P/(\gamma - 1) + \frac{1}{2}B_i B^i$ the energy, and γ the ratio of specific heats. We will study one-dimensional MHD in Cartesian coordinates. All quantities are only dependent on x and t. Note however we must include all three components of the magnetic field. These assumptions reduce the MHD equations to

$$\rho_t + (\rho u)_x = 0 \tag{4.18}$$

$$(\rho u)_t + (\rho u^2 + P^*)_x = 0 \tag{4.19}$$

$$(\rho v)_t + (\rho u v - B_x B_y)_x = 0 \tag{4.20}$$

$$(\rho w)_t + (\rho u w - B_x B_z)_x = 0 \tag{4.21}$$

$$(B_y)_t + (B_y u - B_x v)_x = 0 \tag{4.22}$$

$$(B_z)_t + (B_z u - B_x w)_x = 0 \tag{4.23}$$

$$E_t + ((E + P^*)u - B_x(B_x u + B_y v + B_z w))_x = 0 \tag{4.24}$$

where we use $\mathbf{u} = (u, v, w)$ as the components of u_i. Note that because of our assumption that there is only one spacial dimension, B_x is constant by virtue of the no monopole constraint: $\partial_i B^i = 0$.

The resulting Jacobian of (4.19)–(4.24), is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\gamma-3}{2}u^2 + \frac{\gamma-1}{2}(v^2+w^2) & (3-\gamma)u & (1-\gamma)v & (1-\gamma)w & (2-\gamma)B_y & (2-\gamma)B_z & \gamma-1 \\ -uv & v & u & 0 & -B_x & 0 & 0 \\ -uw & w & 0 & u & 0 & -B_x & 0 \\ -B_yu/\rho + B_xv/\rho & B_y/\rho & -B_x/\rho & 0 & u & 0 & 0 \\ -B_zu/\rho + B_xw/\rho & B_z/\rho & 0 & -B_x/\rho & 0 & u & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \end{pmatrix} \tag{4.25}$$

where we make the definitions $H = (E + P^*)/\rho$ and

$$\alpha_1 = -u\left(H\frac{\gamma-1}{2}\mathbf{u}^2 + \frac{B_x}{\rho}\mathbf{B}\cdot\mathbf{u}\right) \tag{4.26}$$

$$\alpha_2 = H - \frac{B_x^2}{\rho} - (\gamma-1)u^2 \tag{4.27}$$

$$\alpha_3 = (1-\gamma)uv - B_xB_y/\rho \tag{4.28}$$

$$\alpha_4 = (1-\gamma)uw - B_xB_z/\rho \tag{4.29}$$

$$\alpha_5 = (2-\gamma)B_yu - B_xv \tag{4.30}$$

$$\alpha_6 = (2-\gamma)B_zu - B_xw \tag{4.31}$$

$$\alpha_7 = \gamma u. \tag{4.32}$$

We now proceed to compute the eigenvalues and eigenvectors of A.

4.4.1 Jacobian Eigenvalues

We are interested in the eigenvalues λ_n given by $A\mathbf{x} = \lambda_n\mathbf{x}$ with eigenvectors $\mathbf{x} \in \mathbb{R}^7$. A straightforward cofactor expansion calculation leads to the result

$$\lambda_1 = u - c_f, \quad \lambda_2 = u - c_a, \quad \lambda_3 = u - c_s, \quad \lambda_4 = u, \quad \lambda_5 = u + c_s, \quad \lambda_6 = u + c_a, \quad \lambda_7 = u + c_f \tag{4.33}$$

where $\lambda_i \leq \lambda_{i+1}$ and c_f, c_a, c_s are known as the fast, Alfvén, and slow characteristic speeds respectively. They are given by

$$c_a^2 = b_x^2, \quad c_{f,s}^2 = [(a^*)^2 \pm \sqrt{(a^*)^4 - 4a^2 b_x^2}]/2 \tag{4.34}$$

where we have defined

$$b_i = B_i/\sqrt{\rho}, \quad b^2 = b_x^2 + b_y^2 + b_z^2, \quad (a^*)^2 = (\gamma\rho + B^2)/\rho \tag{4.35}$$

and a is the speed of sound given by $a^2 = \gamma P/\rho$.

4.4.2 Eigenvalue Degeneracy

We seek to establish conditions for the eigenvalues to be distinct. By inspection of the definitions of c_a and $c_{f,s}$ we see there are two natural cases to consider:

Case 1: $B_x = 0$. If this condition is satisfied, we have $c_s = c_a = 0$.

Case 2: $B_y^2 + B_z^2 = 0$, $c_f^2 = \max\{a^2, b_x^2\}$, and $c_s^2 = \min\{a^2, b_x^2\}$.

In Case 1, we see that the Alfvén and slow eigenvalues are equal to the entropy eigenvalue which are are all identically u. Hence, λ_1, λ_7, and λ_4 are the distinct eigenvalues, where λ_4 has a multiplicity of 5.

Case 2 breaks into two subcases.

Subcase 2a: Because $|a| = |b_x|$ we find $c_f^2 = c_s^2$. Since $\max\{a^2, b_x^2\} = b_x^2 = a^2$, then $c_f^2 = c_s^2 = c_a^2$ implies $u \pm c_a$ has multiplicity 3.

Subcase 2b: $a^2 \neq b^2$. In this case either $c_f^2 = b_x^2$ or $c_s^2 = b_x^2$. Thus $u \pm c_a$ has multiplicity 2.

4.4.3 Eigenvector Degeneracy

We now consider the degeneracy problem for the eigenvectors of A. Recall that we require the eigenvectors to be complete for our Roe solver to work properly. The eigenvectors corresponding to λ_i are given in [24]. We require the following definitions to simplify the form of the eigenvectors

$$c = \{c_f, c_s\}, \quad g = \mp(B_z v \mp B_y w)\text{sgn}(B_x) \tag{4.36}$$

$$h = \frac{c^2}{\gamma - 1} \pm cu \mp \frac{B_x c(B_y v + B_z w)}{\rho(c^2 - b_x^2)} + \frac{\gamma - 2}{\gamma - 1}(c^2 - a^2). \tag{4.37}$$

where sgn is the sign function. The right eigenvectors of A are given by

$$R_{u\pm c} = \begin{pmatrix} 1 \\ u \pm c \\ v \mp \frac{B_x B_y c}{\rho(c^2 - b_x^2)} \\ w \mp \frac{B_x B_z c}{\rho(c^2 - b_x^2)} \\ \frac{B_y c^2}{\rho(c^2 - b_x^2)} \\ \frac{B_z c^2}{\rho(c^2 - b_x^2)} \\ \frac{u^2+v^2+w^2}{2} + h \end{pmatrix} \quad R_{u\pm c_a} = \begin{pmatrix} 0 \\ 0 \\ \mp B_z \text{sgn}(B_x) \\ \pm B_y \text{sgn}(B_x) \\ B_z/\sqrt{\rho} \\ -B_y/\sqrt{\rho} \\ g \end{pmatrix} \quad R_u = \begin{pmatrix} 1 \\ u \\ v \\ w \\ 0 \\ a \\ \frac{u^2+v^2+w^2}{2} \end{pmatrix}. \tag{4.38}$$

For the degeneracy problem, we consider the same two cases as in the previous section. For Case 1, we define the eigenvectors via limits and need to establish two identities to simplify this task. First we compute

$$c_s^2 c_f^2 = \frac{1}{4}\left((a^*)^4 - (a^*)^4 + 4a^2 b_x^2\right) \tag{4.39}$$

$$= a^2 b_x^2 \tag{4.40}$$

$$(c_s^2 - b_x^2)(c_f^2 - b_x^2) = \left(\frac{1}{2}(a^*)^2 - b_x^2\right)^2 - \frac{1}{4}\left((a^*)^2 - b_x^2\right) \tag{4.41}$$

$$= -b_x^2(b_y^2 + b_z^2). \tag{4.42}$$

Together with

$$c_s = \frac{a|b_x|}{c_f}, \quad c_s^2 - b_x^2 = -\frac{b_x^2(b_y^2 + b_z^2)}{c_f^2 - b_x^2}. \tag{4.43}$$

These induce the relations

$$\frac{b_y b_x c_s}{c_s^2 - b_x^2} = -\frac{a(c_f^2 - b_x^2)}{c_f(b_y^2 + b_z^2)} \operatorname{sgn}(B_x) b_y \tag{4.44}$$

$$\frac{b_y c_s^2}{\sqrt{\rho}(c_s^2 - b_x^2)} = -\frac{a^2 b_y (c_f^2 - b_x^2)}{c_f \sqrt{\rho}(b_y^2 + b_z^2)}. \tag{4.45}$$

We seek to control the singular behavior of the eigenvectors with components that have factors $B_x c_s/(c_s^2 - b_x^2)$ and $c_s^2/(c_s^2 - b_x^2)$. Since we have singularity issues when $b_x \to c_s$, we take $\lim_{B_x \to 0} \operatorname{sgn}(B_x) = 1$ and consider

$$\lim_{B_x \to 0} \frac{b_y b_x c_s}{(c_s^2 - b_x^2)} = \lim_{B_x \to 0} \frac{B_x B_y c_s}{c_s^2 - B_x^2/\rho} = 0 \tag{4.46}$$

$$\lim_{B_x \to 0} \frac{b_y c_s^2}{\rho^{1/2}(c_s^2 - b_x^2)} = -\lim_{B_x \to 0} \frac{a^2}{c_f^2} \frac{b_y}{\sqrt{\rho}} \frac{c_f^2 - b_x^2}{b_y^2 + b_z^2} = -\frac{a^2 b_y}{\sqrt{\rho}(b_y^2 + b_z^2)}. \tag{4.47}$$

We thus see multiplication by the above mentioned factors regularizes the previously infinite eigenvector components.

In Case 2, one of $\frac{B_y}{c^2 - b_x^2}$, $\frac{B_z}{c^2 - b_x^2}$ will become singular, and neither is defined where $b_y^2 + b_z^2 = 0$ for the fast eigenvectors if $a^2 < b_x^2$ or the slow eigenvectors if $a^2 > b_x^2$.

Moreover, the determinant of the eigenvector matrix is proportional to $(c_f^2 - c_s^2)^2$ and $(c_f^2 - c_s^2) \to 0$ as $b_y^2 + b_z^2 \to 0$ and $a^2 - b_x^2 \to 0$. This is the singularity structure we need to eliminate. Note how the singularities in this case are due to the vanishing of the magnetic field tangential to the direction of fluid flow. This is in contrast to Case 1 where the singularities arose due to the vanishing of the fluid flow in the x direction.

We make the definitions

$$\alpha_f = \frac{\sqrt{c_f^2 - b_x^2}}{\sqrt{c_f^2 - c_s^2}} \quad \alpha_s = \frac{1}{c_f}\frac{\sqrt{c_f^2 - a^2}}{\sqrt{c_f^2 - c_s^2}} \quad \beta_y = \frac{B_y}{b_y^2 + b_z^2} \quad \beta_z = \frac{B_z}{\sqrt{b_y^2 + b_z^2}} \tag{4.48}$$

and note the identity

$$\frac{b_y^2 + b_z^2}{\alpha_f^2 \alpha_s^2} = c_f^2 - c_s^2 \tag{4.49}$$

which shows the relationship between vanishing quantities. Scaling the fast, slow and Alfvén eigenvectors by α_f, α_s, and $(b_y^2 + b_z^2)^{-1/2}$ respectively, removes the singularities of the eigenvector matrix. Hence this procedure eliminates degeneracy problems thereby producing a complete set of eigenvectors for a Roe solver.

4.5 The GRMHD Eigenvalue Problem

We now summarize a version of the GRMHD equations due to [20]. Define primitive variables $\mathcal{P} = (\rho_0, v^j, \epsilon, B^j)^T$ where ρ_0 is the rest mass density, v^j is the coordinate velocity of the fluid, ϵ is the specific internal energy of the fluid, and B^j is the magnetic field in the frame of fiducial observers moving with four velocity $n^a = (1, -\beta)/\alpha$ where

α is the lapse and β is the shift[1]. We define the fluid enthalpy by $h_e = \rho_0(1+\epsilon)+P$ where P is the pressure and will be given by an equation of state that we need not specify. We also define various fluid related quantities

$$\chi = \frac{\partial P}{\partial \rho_0}, \quad \kappa = \frac{\partial P}{\partial \epsilon}\gamma, \quad \frac{\partial h_e}{\partial \rho_0} = 1+\epsilon+\chi, \quad h_e c_s^2 = \rho_0 \chi + \frac{P}{\rho_0}\kappa \tag{4.50}$$

where c^s is the speed of sound. The GRMHD equations may be written in balance law form

$$\frac{\partial F^0}{\partial t} + \frac{\partial F^j}{\partial x^j} = S \tag{4.51}$$

where F^0 and the three vectors F^j are dependent on quantities involving the primitive variables. The definition of S is not important while F^0 and F^k are given by

$$F^0 = \sqrt{h}\begin{pmatrix} W\rho_0 \\ (h_e W^2 + B^2)v_i - (Bv)B_i \\ h_e W^2 + B^2 - \frac{1}{2}[(Bv)^2 + B^2/W^2] - W\rho_0 - P \\ B^i \end{pmatrix} \tag{4.52}$$

$$\frac{F^j}{\alpha\sqrt{h}} = \begin{pmatrix} W\rho_0\overline{v}^j \\ ([h_e W^2 + B^2]v_i - (Bv)B_i)\overline{v}^j + h_i^j(P + \frac{1}{2}[(Bv)^2 + B^2/W^2]) - (B_i/W^2 + (Bv)v_i)B^j \\ (h_e W^2 + B^2 - \frac{1}{2}[(Bv)^2 + B^2/W^2] - W\rho_0 - P)\overline{v}^j + (P + \frac{1}{2}[(Bv)^2 + B^2/W^2])v^j - (Bv)B^j \\ B^i\overline{v}^j - B^j(v^i - \beta^i/\alpha) \end{pmatrix} \tag{4.53}$$

where we have defined $\overline{v}^j = v^j - \beta^j/\alpha$, a Lorentz factor $W = 1/(1-[v_i v^i]^2)$, and $h = \det(h_{ij})$ where h_{ij} is the metric on the spacial slices. Because we ignore the source

[1] n^a gives the trajectory of the observers moving normal to the two-manifold ADM foliation, i.e. the spacial slices

for the following calculation, we set it to zero. We define $\mathcal{F}^a$ using the conservative form of the above equation written in covariant form.

$$0 = \frac{\partial F^a(\mathcal{P})}{\partial x^a} = \frac{\partial F^a(\mathcal{P})}{\partial \mathcal{P}} \frac{\partial \mathcal{P}}{\partial x^a} = \mathcal{F}^a \frac{\partial \mathcal{P}}{\partial x^a}. \tag{4.54}$$

We note the system will be hyperbolic if the determinant of the Jacobian contracted with a unit timelike vector is nontrivial. Thus the eigenvalue problem will yield real eigenvalues and a complete set of eigenvectors. We now fix a direction x^k and consider the eigenvalue problem

$$\det[\mathcal{F}^0(-\lambda^k) + \mathcal{F}^k] = 0 \tag{4.55}$$

where λ^k is one eigenvalue in the x^k direction. There is no loss of generality in choosing this direction; it could represent any of the directions. Differentiation of F^a with respect to the primitive variables yields

$$\mathcal{F}^0 = \sqrt{h}\begin{pmatrix} W & W^3\rho_0 v_j & 0 & 0_j \\ W^2\gamma v_i & h_{ij}Q + 2h_e W^4 v_i v_j - B_i B_j & (\rho_0+\kappa)W^2 v_i & 2v_i B_j - B_i v_j - (Bv)h_{ij} \\ \gamma W^2 - W - \chi & (2h_e W^4 + B^2 - W^3\rho_0)v_j - (Bv)B_j & (\rho_0+\kappa)W^2 - \kappa & (2 - 1/W^2)B_j - (Bv)v)j \\ 0^i & 0^i_j & 0^i & h^i_j \end{pmatrix} \tag{4.56}$$

$$\mathcal{F}^k = \alpha\sqrt{h}\begin{pmatrix} W\overline{v}^k & W\rho_0(W^2 v_j \overline{v}^k + h^k_j) & 0^k & 0^k_j \\ W^2\gamma v_i \overline{v}^k + h^k_i \chi & \mathcal{A}^k_{ij} & (\rho_0+\kappa)W^2 v_i \overline{v}^k + h^k_i \kappa & \mathcal{B}^k_{ij} \\ (W^2\gamma - W - \chi)\overline{v}^k + \chi v^k & \mathcal{C}^k_j & (\rho_0+\kappa)W^2\overline{v}^k - \kappa(\overline{v}^k - v^k) & \mathcal{D}^k_j \\ 0^{ik} & B^i h^k_j - B^k h^i_j & 0^{ik} & h^i_j \overline{v}^k - h^k_j \overline{v}^i \end{pmatrix} \tag{4.57}$$

where we define

$$\mathcal{A}^k_{ij} = (h_{ij}Q + 2h_e W^4 v_i v_j - B_i B_j)\overline{v}^k + (Qv_i - (Bv)B_i)h^k_j + h^k_i((Bv)B_j - B^2 v_j)$$
$$- (h_{ij}(Bv) + v_i B_j - 2B_i v_j)B^k \tag{4.58}$$

$$\mathcal{B}^k_{ij} = (2v_i B_j - B_i v_j - (Bv)h_{ij})\overline{v}^k - (h_{ij}/W^2 + v_i v_j)B^k \tag{4.59}$$

$$\mathcal{C}^k_j = (2hW^4 v_j - (Bv)B_j + B^2 v_j - \rho_0 W^3 v_j)\overline{v}^k + (Q - W\rho_0)h^k_j + ((Bv)B_j - B^2 v_j)v^k - B_j B^k \tag{4.60}$$

$$\mathcal{D}^k_j = (2B_j - (Bv)v_j - B_j/W^2)\overline{v}^k + ((Bv)v_j + B_j/W^2)v^k - v_j B^k - (Bv)h^k_j. \tag{4.61}$$

where $0^i = 0^j = (0, 0, 0)$ where i indexes rows and j indexes of columns.

4.5.1 Eigenvalue Problem

Through an extensive yet straightforward cofactor expansion it can be shown

$$\det(\mathcal{F}^k - \lambda^k \mathcal{F}^0) = -\alpha^8 h^5 \rho_0 h_e W^3 \lambda^k (\overline{v}^k - \lambda^k)\Delta^{kk} \Big\{ h_e W^4 (1 - c_s^2)(\overline{v}^k - \lambda^k)^4$$
$$+ \big[(\overline{v}^k - \lambda^k)^2 (h_e W^2 c_s^2 + B^2 + W^2 (Bv)^2) - c_s^2 (W(Bv)(\overline{v}^k - \lambda^k) + B^k/W)^2\big]\big[((\overline{v}^k - \lambda^k) - v^k)^2 - h^{kk}\big]\Big\} \tag{4.62}$$

where

$$\Delta^{kk} = \left[(\bar{v}^k - \lambda^k)^2 Q - 2(\bar{v}^k - \lambda^k)(Bv)B^k - \frac{1}{W^2} B^k B^k \right] \tag{4.63}$$

and $Q = h_e W^2 + B^2$. The zeros of (4.62) give the desired eigenvalues. There is one trivial eigenvalue $\lambda^k = 0$, one associated with the entropy wave $\overline{v}^k - \lambda^k = 0$, two for the Alfvén waves given by the solution of $\Delta^{kk} = 0$ and four magnetosonic wave eigenvalues given by the solutions to the quartic.

4.5.2 Eigenvector Problem

We wish to compute the seven eigenvectors corresponding to the entropy wave, two Alfvén waves, and the four magnetosonic waves. We will represent eigenvectors as $(e^0, e^i, e^4, \hat{e}^i)$ where i ranges over $i \in \{1, 2, k\}$ for some fixed direction x^k. We define $a^k = \bar{v}^k - \lambda^k$ and compute the eigenvectors for the GRMHD equations by solving the system [20]

$$0 = e^0 a^k + (ve)W^2\rho_0 a^k + e^k\rho_0 \tag{4.64}$$

$$0 = e^0[(W^2\gamma - \chi)a^k + \chi v^k] + e^4[(\rho_0 + \kappa)W^2 a^k + \kappa(v^k - a^k)] \tag{4.65}$$

$$+ (ve)[2hW^4 a^k - C^k] + (Be)D^k + e^k[hW^2 - E^k/a^k] \tag{4.66}$$

$$0 = (\chi e^0 + \kappa e^4)(v^k - a^k + W^2 a^k)W^{-2} + (ve)[hW^2 a^k - C^k W^{-2}] \tag{4.67}$$

$$+ (Be)D^k W^{-2} - e^k E^k/(W^2 a^k) \tag{4.68}$$

$$0 = (\chi e^0 + \kappa e^4)(B^k - (Bv)(v^k - a^k)) + (ve)(Bv)C^k \tag{4.69}$$

$$+ (Be)[hW^2 a^k - D^k(Bv)] + e^k(Bv)E^k/a^k \tag{4.70}$$

where $a^k = \bar{v}^k - \lambda^k$ for λ^k an eigenvalue. (Bv) denotes the inner product of B and v. We note that we may immediately combine these equations to find

$$e^4 = e^0 P/\rho_0^2 \tag{4.71}$$

In the case of the entropy wave where $a^k = 0$, a straightforward calculation shows the system implies

$$e = c(1, 0^i, -\chi/\kappa, 0^i)^T \tag{4.72}$$

In the case of the Alfvén eigenvectors where a^k is a solution of $\Delta^{kk} = 0$ we can solve the system. Allowing $\Delta^{kk} = 0$ the eigenvector equations show the Alfvén eigenvector components take the form

$$e^0 = 0 \tag{4.73}$$

$$e^1 = \frac{ve}{v_1 B_2 - v_2 B_1}\left[B_2(W^2(Bv)\frac{a^k}{B^k} + 1) - v_2 W^2\left(B^2\frac{a^k}{B^k} - (Bv)\right)\right] - (ve)W^2 B^1 \frac{a^k}{B^k} \tag{4.74}$$

$$e^2 = \frac{ve}{v_1 B_2 - v_2 B_1}\left[-B_1(W^2(Bv)\frac{a^k}{B^k} + 1) + v_1 W^2\left(B^2\frac{a^k}{B^k} - (Bv)\right)\right] - (ve)W^2 B^2 \frac{a^k}{B^k} \tag{4.75}$$

$$e^k = -(ve)W^2 a^k \tag{4.76}$$

$$e^4 = 0 \tag{4.77}$$

$$\hat{e}^1 = \frac{ve}{v_1 B_2 - v_2 B_1}\left[B_2(W^2(Bv)\frac{a^k}{B^k} + 1) - v_2 W^2\left(B^2\frac{a^k}{B^k} - (Bv)\right)\right] \tag{4.78}$$

$$\hat{e}^2 = \frac{ve}{v_1 B_2 - v_2 B_1}\left[-B_1(W^2(Bv)\frac{a^k}{B^k} + 1) + v_1 W^2\left(B^2\frac{a^k}{B^k} - (Bv)\right)\right] \tag{4.79}$$

$$\hat{e}^k = 0 \tag{4.80}$$

We now turn our attention to the magnetosonic eigenvectors where a^k is a solution to the quartic in equation (4.62). In this case, $\Delta^{kk} \neq 0$ the system

$$\begin{pmatrix} hW^4a^k - C^k + W^2E^k & D^k \\ hW^4(Bv)a^k & hW^2a^k \end{pmatrix} \begin{pmatrix} (ve) \\ (Be) \end{pmatrix} = -e^0 \begin{pmatrix} \frac{hc_s^2}{\rho_0}(v^k - a^k + W^2a^k) + \frac{E^k}{\rho_0} \\ \frac{hc_s^2}{\rho_0}((Bv)W^2a^k + B^k) \end{pmatrix} \tag{4.81}$$

can be inverted to find (ve) and (Be) given by

$$(ve) = -\frac{e^0}{\rho_0}\frac{h}{\Delta^{kk}}\frac{1}{a^k}\Big\{(v^k - a^k)\left[hc_s^2W^2(a^k)^2 - c_s^2((Bv)Wa^k + \frac{B^k}{W})^2 + (a^k)^2(|B|^2 + W^2(Bv)^2)\right]$$
$$+ hc_s^2W^4(A^k)^3 - B^kW(a^k(Bv)W + B^k/W)a^kc_s^2 + W^2(a^k)^2(a^k|B|^2 - (Bv)B^k)\Big\} \equiv e^0\psi^k \tag{4.82}$$

$$(Be) = [\Delta^{kk} + a^k(Bv)B^k][c_s^2W^3(W(Bv)a^k + B^k/W) - W^4(Bv)a^k] - W^2(Bv)a^kB^kB^k \equiv \xi^k \tag{4.83}$$

which when combined with the initial system gives

$$e^k = -\rho_0^{-1}(e^0a^k + (ve)W^2\rho_0a^k) = -e^0a^k\rho_0^{-1}(1 + \psi^kW^2\rho_0). \tag{4.84}$$

Demanding $i \neq k$ and recalling $\hat{e}^k = 0$ we have

$$0 = B^ie^k - B^ke^i + a^k\hat{e}^i. \tag{4.85}$$

We can combine the eigenvector equations to show

$$0 = [W^2\gamma v_ia^k + \chi h_i^k]e^0 + \Big\{e_i[Qa^k - (Bv)B^k] + (ve)[2hW^4v_ia^k - h_i^k|B|^2 + 2B_iB^k]$$
$$+ (Be)[-B_ia^k + h_i^k(Bv) - v_iB^k] + e^k[Qv_i - (Bv)B_i]\Big\}e^4 + [(\rho_0 + \kappa)W^2v_ia^k + h_i^k\kappa]e^4$$

$$+\left\{\hat{e}_i\left[-(Bv)a^k - \frac{B^k}{W^2}\right] + (v\hat{e})\left[-B_i a^k - v_i B^k + h_i^k(Bv)\right] + (B\hat{e})\left[2v_i a^k + \frac{h_i^k}{W^2}\right]\right.$$
$$\left. - \hat{e}^k\left[(Bv)v_i + \frac{B_i}{W^2}\right]\right\} \tag{4.86}$$

which we write in the form

$$0 = \alpha_i^k e^0 + e_i\beta^k + \gamma^k\hat{e}_i + \xi_i^k. \tag{4.87}$$

Solving for the eigenvector components, we compute

$$e^i = -\frac{\alpha_i^k a^k e^0 - B^i e^k\gamma^k + a^k\xi_i^k}{a^k\beta^k + B^k\gamma^k} \tag{4.88}$$

$$\hat{e}^i = -\frac{\alpha_i^k B^k e^0 + \beta^k B^i e^k + B^k\xi_i^k}{a^k\beta^k + B^k\gamma^k}. \tag{4.89}$$

which determines the GRMHD eigenvalues. Simplifying the denominator, we find

$$a^k\beta^k + B^k\gamma^k = Q(a^k)^2 - 2a^k(Bv)B^k - W^{-2}B^kB^k = \Delta^{kk} \tag{4.90}$$

which is the same factor as in the determinant of the GRMHD equations. Thus we see we have singularity problems when either $v_1B_2 - v_2B_1 = 0$, $B^k = 0$, or $\Delta^{kk} = 0$. Generalizing the analysis of [4], we seek conditions when the eigenvalues of the GRMHD equations are unique. We first note that when the discriminant of Δ^{kk} vanishes, we have degeneracy. This is equivalent either $(Bv)^2 + QW^{-2} = 0$ or $B^k = 0$. If the first condition is met, we have essentially trivialized the GRMHD equations. The second condition has already been taken into account. Thus, we do not expect to have singularity problems with the Alfvén eigenvectors. The magnetosonic case is slightly more difficult. We consider a theorem from [27] which requires:

Definition: Let F be a field of characteristic 0 and $f(x) \in F[x]$ a polynomial of degree n. Represent

$$f = \prod_{i=1}^{n} (x - \alpha_i) \tag{4.91}$$

and define

$$\Delta = \prod_{i<j} (\alpha_i - \alpha_j). \tag{4.92}$$

Then the discriminant of $f(x) \in F[x]$ is $D = \Delta^2$.

Theorem: $f(x)$ has repeated roots iff $D = 0$.

Thus we have rephrased our problem in terms of the vanishing of D. The following corollary is useful for considering our specific quartic.

Corollary: Let $\lambda_4, \lambda_5, \lambda_6, \lambda_7$ be the roots of a 4-th order polynomial over the reals. Then

$$\Delta^2 = D = \left(c \prod_{i<j} (\lambda_i - \lambda_j) \right)^2 = c^2(\lambda_4 - \lambda_7)^2(\lambda_5 - \lambda_7)^2(\lambda_6 - \lambda_7)^2(\lambda_4 - \lambda_6)^2(\lambda_5 - \lambda_6)^2(\lambda_4 - \lambda_5)^2 \tag{4.93}$$

An alternative expression for D in terms of roots of $f = \sum_i^4 c_i x^i$ is given by

$$\begin{aligned} D = \Big[&(c_1^2 c_2^2 c_3^2 - 4c_1^3 c_3^3 - 4c_1^3 c_2^3 c_4 + 18c_1^3 c_2 c_3 c_4 - 27c_1^4 c_4^2 + 256 c_0^3 c_4^3) \\ &+ c_0(-4c_2^3 c_3^2 + 18 c_1 c_2 c_3^3 + 16 c_2^4 c_4 - 80 c_1 c_2^2 c_3 c_4 - 6c_1^2 c_3^2 c_4 + 144 c_1^2 c_2 c_4^2) \\ &c_0^2(-27c_3^4 + 144 c_2 c_3^2 c_4 - 128 c_2^2 c_4^2 - 192 c_1 c_3 c_4^2) \Big]. \end{aligned} \tag{4.94}$$

which in principal completely determines degeneracy of the quartic associated to the eigenvalues of the magnetosonic waves.

4.6 Conclusions

We have calculated all the eigenvalues and eigenvectors of a balance law form of the GRMHD equations and established singularity and degeneracy conditions. Multiplication of the eigenvector matrix by $(\Delta^{kk})^4(v_1B_2 - v_2B_1)^2(B^k)^2$ will resolve the singularity problems in the spirit of [4]. This will force the GRMHD equations to be strongly hyperbolic.

Chapter 5

The Axisymmetric Initial Value Problem for GRMHD

In the previous chapter we consider aspects of the characteristic problem for GRMHD. An additional problem for attempting to simulate the set of equations it to construct appropriate initial data for a simulation. In particular, one model we are interested in is modeling a differentially rotating magnetized neutron star. This is impossible without having an initial configuration for the system. We will describe a formalism which allows us to make steps in solving this problem. Solving this problem is necessary to consider solving the full time evolution problem of the system.

This requires evolving an initial metric coupled to a specified matter distribution with the Einstein equations. One cannot choose the initial metric arbitrarily, but must demand it satisfies the constraint equations of general relativity at a fixed time. This is analogous to the requirement in electrodynamics that one must choose the initial electric and magnetic fields in any dynamic configuration in a manner consistent with the time independent electrodynamic equations.

We suppose a stationary axisymmetric spacetime. Equivalently, we assume the existence of two Killing vectors , one timelike, and another spacelike with closed orbits. We will consider two types of these spacetimes. First, we reproduce current results on spacetimes that obey the circularity condition. This restricts our fluid matter's propagation to planes parallel to the equatorial plane of a magnetized neutron star and the magnetic field may be purely poloidal. This assumption of circularity is frequently made in order to simplify the form of the spacetime metric. Such a metric is convenient for calculation but is often an unphysical restriction of the matter content of an axisymmetric spacetime. One example of its use is [5] to model nonmagnetized differentially rotating stars.

In order to generalize the possible dynamics of our matter, we remove the condition of circularity and consider the initial value problem for non-circular spacetimes. In [3], uniformly rotating magnetized neutron stars were considered. We extend this work by deriving field equations and equations of motion for a differentially rotating magnetized neutron star. We state the equations for the general initial value problem (without assuming circularity) and leave numerical computations for future work.

5.1 Introduction

We are ultimately interested in a general stationary axisymmetic spacetimes. A spacetime is stationary if it admits a timelike Killing vector τ^μ. A spacetime is axisymmetric if it admits a compact spacelike Killing vector field ξ^μ. A spacetime is stationary and axisymmetric if it is stationary, axisymmetric and the Killing vectors satisfy the orthogonality relation: $[\tau, \xi] = 0$. Killing vectors generate spacetime isometries via the exponential map. Thus the components of the spacetime metric will not depend on coordinates which are adapted to the Killing vectors. If one assumes a stationary and axisymmetric spacetime, one may assume a metric of the form

$$ds^2 = g_{\mu\nu}(x^2, x^3)dx^\mu dx^\nu \tag{5.1}$$

with coordinate vector $x = (t, x^2, x^3, \phi)$ where $\tau = \partial_t$ and $\xi = \partial_\phi$.

Now consider the following theorem [30, p.163]

Theorem 1. *Let ξ^a and ψ^a be two commuting Killing vector fields that satisfy*

(i) $\xi_{[a}\psi_b\nabla_c\xi_{d]}$ and $\xi_{[a}\psi_b\nabla_c\psi_{d]}$ each vanish at least at one point.

(ii) $\xi^a R_a{}^{[b}\xi^c\psi^{d]} = \psi^a R_a{}^{[b}\xi^c\psi^{d]} = 0$

We call (ii) the circularity condition. Then the 2-planes orthogonal to ξ^a and ψ^a are integrable, i.e. they define 2-manifolds that foliate the full spacetime.

It can be shown that if this theorem holds and one chooses x_2 and x_3 wisely, a static axisymmetric metric can be written in the form

$$ds^2 = -V(dt - \omega d\phi)^2 + V^{-1}[\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 + dz^2)] \tag{5.2}$$

where V, ω, and γ are only functions of ρ and z. There are only two nontrivial Einstein equations for this metric. This is a dramatic simplification from a full axisymmetric problem and is the starting point for the majority of standard axisymmetric analyses. The hypotheses of the above theorem are satisfied for many spacetimes. Note that the theorem as stated is a result from differential geometry. When the Einstein equations are considered in conjunction with the theorem, the conditions of the theorem become conditions on the stress energy tensor. The MHD stress tensor does not satisfy these conditions in general and hence one my not apply the circularity condition for GRMHD. For instance, if one wishes to model a differentially rotating magnetized neutron star where the fluid velocity is not restricted to the equatorial plane, one must relax the conditions of the theorem and only assume the existence of the two Killing vectors. The simple form of equation (5.2) thus does not hold.

As a result of the above argument the Einstein equations for of a stationary axisymmetric spacetime coupled to MHD are much more involved because the metric is significantly more complicated. Nevertheless, we wish to solve the corresponding initial value problem for GRMHD. One aspect of this problem can best be explained in terms of an analogy with electrodynamics [30, ch.10]. Recall that the vacuum Maxwell equations are given by

$$\epsilon_0 \partial_t \mathbf{E} = \frac{1}{\mu_0} \nabla \times \mathbf{B} - \mathbf{J}, \quad \partial_t \mathbf{B} = -\nabla \times \mathbf{E} \tag{5.3}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \rho / \epsilon_0. \tag{5.4}$$

The Maxwell equations govern the time evolution of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ given a charge distribution ρ and a current density $\mathbf{J}$. Equations (5.3) govern the time evolution of the electric and magnetic fields, and equations (5.4) are constraint equations that both fields must satisfy for all time. Thus one must specify initial data for the fields that is consistent with the constraint equations. This requires solving the time independent electrodynamics equations to generate valid initial data.

We can formulate a similar nonlinear initial value problem in general relativity. To do this we to assume our spacetime admits a timelike vector field that foliates the full four dimensional spacetime into spacelike three-manifolds. We can prescribe initial data on one of the three manifolds whose time evolution is governed by the dynamic Einstein equations. Again we cannot specify this initial data arbitrarily, but much choose an initial metric and matter fields to satisfy the elliptic (constraint) Einstein equations. This is the initial value problem for general relativity. We will further restrict to axisymmetric spacetimes.

5.2 Axisymmetry with Circularity

The key assumption that simplifies the form of the metric in axisymmetric spacetimes is the previously mentioned theorem. We state an similar theorem that has does not involve the circularity condition:

Theorem 2. *(Frobenius Theorem) Let X^a and N^a be two vectors on a Lorentzian manifold. If*

$$\nabla_a X_{[b} X_c N_{d]} = 0, \quad \text{and,} \quad \nabla_a N_{[b} N_c X_{d]} = 0$$

then X^a and N^a define integrable 2-manifolds that foliate for the full Lorentzian manifold.

We note that this theorem has the same result as our previously mentioned theorem. We make the assumption of the Frobenius Theorem and the circularity and consider the resulting Einstein's equations.

5.2.1 Fluid Calculations

Unmagnetized differentially rotating stars are studied in the context of stationary axisymmetric general relativity [5]. The solutions that we are considering are equilibrium configurations and are in a sense generalizations of gravitating Newtonian spheroids. The authors suppose a metric of the form

$$ds^2 = -e^{\gamma+\rho} dt^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2) + e^{\gamma-\rho} r^2 \sin^2\theta (d\phi - \omega dt)^2 \tag{5.5}$$

where ρ, γ, α and ω are only dependent on r and θ. Setting $G = c = 1$, the perfect fluid stress tensor is given by

$$T^{ab} = (\rho_0 + \rho_i + P)u^a u^b + Pg^{ab} \tag{5.6}$$

where ρ_0 is the rest energy density, ρ_i is the internal energy density, P is the pressure, and u^a is the matter four velocity. This is the same metric we previously mentioned in a different representation. We wish to compute the Einstein and matter equations. To this end, we find the velocity components of the fluid by first defining the proper matter velocity by

$$v = (\Omega - \omega)r \sin\theta e^{-\rho} \tag{5.7}$$

where $\Omega \equiv d\phi/dt = (d\phi/d\tau)(d\tau/dt) = u^\phi/u^t$. We assume the fluid four-velocity vector is normed to -1, there exist Killing vectors $\partial_t, \partial_\phi$, and meridonal circulation ($u^r = u^\theta = 0$) which is a result of the circularity condition. With this information, we find all the components of the four velocity in terms of the proper velocity

$$-1 = u^a u_a = g_{tt}(u^t)^2 + 2g_{t\phi}u^t u^\phi + g_{\phi\phi}(u^\phi)^2$$

$$= e^{\gamma-\rho} r^2 \sin^2\theta [u^\phi - \omega u^t]^2 \tag{5.8}$$

from which we substitute out u^ϕ and immediately solve for u^t to find

$$u^a = [e^{(\gamma+\rho)/2}\sqrt{1-v^2}]^{-1}(1, 0, 0, \Omega) \tag{5.9}$$

5.2.2 Matter Equations

Now now consider the matter equations given by the vanishing of the divergence of the stress-energy tensor.

$$0 = \nabla_a T^{ab} = [\rho_0 + \rho_i + P](u^a \nabla_a u^b + u^b \nabla_a u^a) + \nabla^b P \tag{5.10}$$

where we have used the above form of the fluid, the useful identity

$$\nabla_a u^a = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} u^a) = 0. \tag{5.11}$$

and the Killing vector assumptions $\partial_t = \partial_\phi = 0$. After simplification, the above becomes

$$0 = [\rho_0 + \rho_i + P] u^a \nabla_a u_b + \nabla_b P. \tag{5.12}$$

Repeatedly applying our assumptions, we compute

$$u^a \nabla_a u_b = -u^a \Gamma^c_{ab} u_c = -\frac{1}{2} u^a u^d \partial_b g_{ad} = u_a \partial_b u^a$$
$$= (u_t + u_\phi \Omega) \partial_b u^t + u^t u_\phi \partial_b \Omega = -\partial_b (\ln u^t) + u^t u_\phi \partial_b \Omega. \tag{5.13}$$

Thus the matter equations can be expressed in differential form as

$$0 = dP - [\rho_0 + \rho_i + P][d \ln u^t - u^t u_\phi d\Omega] = 0 \tag{5.14}$$

which is called the equation of hydrostatic equilibrium. In the case of circularity, given an equation of state, this can be integrated directly.

5.2.3 The Field Equations

The Einstein equations for the circular axisymmetric metric may be written

$$\nabla^2 [\rho e^{\gamma/2}] = S_\rho (r, \mu) \tag{5.15}$$

$$\left(\nabla^2 + \frac{1}{r}\partial_r - \frac{\mu}{r^2}\partial_\mu\right)[\gamma e^{\gamma/2}] = S_\gamma(r,\mu) \tag{5.16}$$

$$\left(\nabla^2 + \frac{2}{r}\partial_r - \frac{2\mu}{r^2}\partial_\mu\right)[\omega e^{(\gamma-2\rho)/2}] = S_\omega(r,\mu) \tag{5.17}$$

where $\mu = \cos\theta$ and the S_i terms depend on stress tensor and metric terms and there first derivative only (See [5] for the explicit values). Solving these equations together with the equation of hydrostatic equilibrium amount to finding equilibrium configurations of rotating fluid bodies.

In order to solve these equations, we adopt Green's function techniques similar to [5]. We are interested in finding the Green's function for the operators in (5.15)-(5.17). This will allow us to use an iterated Green's function numerical solver to find the metric components in a manner similar to that taken in calculating amplitudes of scattering experiments in quantum mechanics. We seek the general solution to the free space Green's function problem for

$$\left[\nabla^2 + \frac{n}{r}\partial_r - \frac{n\mu}{r^2}\partial_\mu\right]\phi = S \tag{5.18}$$

subject to

$$\lim_{r\to\infty}\phi(r,\mu) = 0 \quad |\phi(0,\mu)| < \infty \tag{5.19}$$

$$\phi(r,1) = \phi(r,-1) \quad \phi_\mu(r,1) = \phi_\mu(r,-1) \tag{5.20}$$

where we have set $\mu = \cos\theta$. We first note that the partial differential operator is currently not in self-adjoint form. Hence, we need to multiply by an appropriate scalar $\rho(r,\mu)$ to make the problem accessible to Green's function techniques.

Recall a linear operator L is self-adjoint on a Hilbert space $\mathcal{H}$ if $L = L^\dagger$ where $(Lu, w) \equiv (u, L^\dagger w)$, where $(\cdot,\cdot)$ is the inner product on $\mathcal{H}$. We take $\mathcal{H}$ to be the space of square Lebesgue integrable functions and define

$$L = \nabla^2 + \frac{n}{r}\partial_r - \frac{n\mu}{r^2}\partial_\mu, \quad \widetilde{L} = \rho L \tag{5.21}$$

and compute

$$\begin{aligned}(\widetilde{L}u, w) &= \int \rho(\nabla^2 u)w + n\int \frac{\rho}{r}u_r w - n\int \frac{\mu\rho}{r^2}u_\mu w \\ &= \int u\left[\nabla^2(\rho w) - n\frac{\partial}{\partial r}\left(\frac{\rho w}{r}\right) + \frac{n}{r^2}\frac{\partial}{\partial \mu}(\mu\rho w)\right] = (u, \widetilde{L}^\dagger w).\end{aligned} \tag{5.22}$$

Hence the self-adjoint condition on ρ becomes

$$\rho\left[\nabla^2 u + \frac{n}{r}u_r - \frac{n\mu}{r^2}u_\mu\right] = \left[\nabla^2(\rho u) - n\frac{\partial}{\partial r}\left(\frac{\rho u}{r}\right) + \frac{n}{r^2}\frac{\partial}{\partial \mu}(\mu\rho u)\right]. \tag{5.23}$$

Recall the Laplacian takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\cot\theta\frac{\partial}{\partial \theta} \tag{5.24}$$

where ϕ dependent terms have been ignored. Setting $\mu = \cos\theta$ we find

$$\frac{\partial}{\partial \theta} = \frac{\partial \mu}{\partial \theta}\frac{\partial f}{\partial \mu} = -\sin\theta\frac{\partial f}{\partial \mu} \tag{5.25}$$

$$\frac{\partial^2}{\partial \theta^2} = (1-\mu^2)f_{\mu\mu} - \mu f_\mu \tag{5.26}$$

from which we note the transformed Laplacian takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} - \frac{2\mu}{r}\frac{\partial}{\partial \mu}. \tag{5.27}$$

Expanding the right hand side of the self-adjoint condition, we find

$$\nabla^2(\rho u) - n\frac{\partial}{\partial r}\left(\frac{\rho u}{r}\right) + \frac{n}{r^2}\frac{\partial}{\partial \mu}(\mu\rho u)$$

$$= \rho\Delta u + \left(2\rho_r - \frac{n}{r}\rho\right) u_r + \left[\frac{1-\mu^2}{r^2} 2\rho_\mu + \frac{n\mu}{r^2}\rho\right] u_\mu + u\rho_{rr} + \frac{2}{r}u\rho_r$$
$$+ \frac{1-\mu^2}{r^2}u\rho_{\mu\mu} - \frac{2\mu}{r^2}\rho_\mu u - \frac{n}{r}\rho_\mu u + 2\frac{n\rho u}{r^2} + \frac{n\mu}{r^2}\rho_\mu u \tag{5.28}$$

We thus require the following are simultaneously satisfied

$$\left(2\rho_r - \frac{n}{r}\rho\right) = \rho\frac{n}{r} \tag{5.29}$$

$$\left[2\frac{1-\mu^2}{r^2}\rho_\mu + \frac{n\mu}{r^2}\rho\right] = -\rho\frac{n\mu}{r^2} \tag{5.30}$$

Thus we find $\rho_r = n\rho/r$ and $\rho_\mu = n\mu\rho/(\mu^2-1)$. Dividing the equations and solving gives $\rho = c_1(\mu)r^n$. Substituting this result back into the previous equation leaves an ordinary differential equation for c_1, which we solve and normalize to determine $\rho = (1-\mu^2)^{n/2}r^n$. Thus $\widetilde{L} = (1-\mu^2)^{n/2}r^n L$ is a self-adjoint operator, and we can state the free space Green's function problem as

$$\rho\left[\nabla^2 + \frac{n}{r}\partial_r - \frac{n\mu}{r^2}\partial_\mu\right] G(r,\mu) = -\frac{4\pi\epsilon}{r^2}\delta(r-r')\delta(\mu-\mu') \tag{5.31}$$

or equivalently,

$$\frac{\partial}{\partial r}\left(r^{n+2}\frac{\partial G}{\partial r}\right) + \frac{r^n}{(1-\mu^2)^{n/2}}\frac{\partial}{\partial \mu}\left[(1-\mu^2)^{1+n/2}\frac{\partial G}{\partial \mu}\right] = -\frac{4\pi\epsilon}{(1-\mu^2)^{n/2}}\delta(r-r')\delta(\mu-\mu') \tag{5.32}$$

for ϵ an arbitrary constant.

We seek to solve the homogenous equation by a separation of variables ansatz. Let $G = R(r)\Theta(\mu)$. Then the equation separates into two ordinary differential equations

$$R'' + \frac{n+2}{r}R' - \frac{\lambda^2}{r^2}R = 0 \tag{5.33}$$

$$(1-\mu^2)\Theta'' - 2\mu(1+n/2)\Theta' + \lambda^2\Theta = 0. \tag{5.34}$$

The Θ equation is a Lengendre type equation. To solve it we use the method of Frobenius. Assume that

$$\Theta = \sum_{i=0}^{\infty} c_i \mu^i \tag{5.35}$$

and substitute into the differential equation to find

$$\begin{aligned} 0 &= \sum_{i=-2}^{\infty}(i+2)(i+1)c_{i+2}\mu^i + \sum_{i=0}^{\infty}[\lambda^2 - i(i-1) - i(n+2)]c_i\mu^i \\ &= \sum_{i=0}^{\infty}[(i+2)(i+1)c_{i+2} + [\lambda^2 - i(i-1) - (n+2)i]c_i]\mu^i. \end{aligned} \tag{5.36}$$

Thus we find the recursion relation

$$c_{i+2} = \frac{i(i-1) + i(n+2) - \lambda^2}{(i+2)(i+1)} c_i. \tag{5.37}$$

We note the series converges iff

$$\lim_{i\to\infty}\left|\frac{c_{i+2}\mu^{i+2}}{c_i\mu^i}\right| < 1. \tag{5.38}$$

In analogy to a similar argument for Legendre polynomials, we must truncate the solution for some i because the series does not converge at $\mu = \pm 1$. This gives the condition

$$i(i-1) + i(n+2) - \lambda^2 = 0 \rightarrow \lambda^2 = i(i+n+1). \tag{5.39}$$

Thus we name the solutions $P_i^{(n)}$, where i denotes the truncation term. We solve the R Euler equation and note the Green's function is given by

$$G = 4\pi\epsilon\left(\frac{r}{r_>}\right)^n \sum_l \frac{r_<^l}{N_{ln}^2(2l+n+1)r_>^{l+1}} P_l^{(n)}(\mu)P_l^{(n)}(\mu')(1-\mu^2)^{n/2} \tag{5.40}$$

where $N_{ln} = \int [P_l^{(n)}]^2$.

5.3 Axisymmetry without the Circularity Condition

We now turn to extending the results of the circular analysis to one for which circularity does not hold. In [3], uniformly rotating magnetized neutron stars are considered. We extend the formalism in the following by calculating the Einstein and matter equations without the assumption of circularity. In this case we no longer have a simple metric. In fact, the metric has no zero components. We will preform a double Kaluza Klein type reduction in two directions of a four dimensional spacetime without assuming hypersurface orthogonality.

Following our assumption of a stationary axisymmetric metric. We consider a Lorentzian four-manifold (M, γ) and decompose γ according to

$$\gamma_{\mu\nu} = \sigma_{\mu\nu} - Q^2 M_\mu M_\nu + s^2 Y_\mu Y_\nu \tag{5.41}$$

where we have made definitions

$$X^\mu = (0,0,0,1) \quad X^\mu X_\mu = s^2 \quad Y^\mu = X^\mu / s^2 \tag{5.42}$$

$$N^\nu = (1,0,0,0) \quad N^\mu N_\mu = -q^2 \tag{5.43}$$

and define an intermediate projection operator

$$g_{\mu\nu} = \gamma_{\mu\nu} - s^2 Y_\nu Y_\mu. \tag{5.44}$$

We note that s and q are functions of local coordinates. Thus we have not completely fixed coordinates associated with the two Killing vectors.

We can think of $g_{\mu\nu}$ as a projection operator defined on the full spacetime which projects onto a three manifold orthogonal to the timelike Killing vector. It can be shown that $g_{\mu\nu}$ is the induced metric on the three manifold and thus can be used to raise and lower indices on purely three manifold tensors. Similarly $\sigma_{\mu\nu}$ projects tensors in the full spacetime onto a two manifold. Next, we define

$$\begin{aligned} {}^3N_\mu &= g^\lambda_\mu N_\lambda = \gamma^\lambda_\mu N_\lambda - s^2 Y_\mu Y^\lambda N_\lambda \\ &= N_\mu - N_\phi Y_\mu = (1, 0, 0, -N_\phi/s^2) \end{aligned} \tag{5.45}$$

with normalization

$${}^3N_\mu {}^3N^\mu = -\left(q^2 + \frac{N_\phi^2}{s^2}\right) \equiv -Q^2 \tag{5.46}$$

where N_ϕ is the last component of N_a. Thus M^a and Y^a are orthogonal vectors by construction. We define $M_\mu = {}^3N_\mu/Q^2$ and find that

$$M_\mu Y^\mu = M^\mu \sigma_{\mu\nu} = Y^\mu \sigma_{\mu\nu} = 0. \tag{5.47}$$

We make the "pseudo-maxwell form" definitions in terms of "pseudo-gauge potentials" to be

$$Z_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu, \quad W_{\mu\nu} = \partial_\mu M_\nu - \partial_\nu M_\mu. \tag{5.48}$$

Note that $Y^\mu Z_{\mu\alpha} = Y^\mu W_{\mu\alpha} = 0$. This means $Z_{\mu\nu}$ lives on the three manifold with metric $g_{\mu\nu}$ and $W_{\mu\nu}$ lies in the two manifold with metric $\sigma_{\mu\nu}$. Next, we compute ${}^4\Gamma$ in terms of the two manifold Christoffel symbols and projection elements to be

$$
\begin{aligned}
{}^4\Gamma^{\lambda}_{\mu\nu} = {}^2\Gamma^{\lambda}_{\mu\nu} - \frac{1}{2}M^{\lambda}\left[\partial_\mu(Q^2M_\nu) + \partial_\nu(Q^2M_\mu) + s^2Y_\mu\partial_\nu(N_\phi/s^2) + s^2Y_\nu\partial_\mu(N_\phi/s^2)\right] \\
+\frac{1}{2}Y^{\lambda}\left[\partial_\mu(s^2Y_\nu) + \partial_\nu(s^2Y_\mu)\right] \\
+\frac{1}{2}\sigma^{\lambda\alpha}\left[s^2(Y_\mu Z_{\nu\alpha} - Y_\nu Z_{\mu\alpha}) - Q^2(M_\mu W_{\nu\alpha} + M_\nu W_{\mu\alpha}) - Y_\mu Y_\nu\partial(s^2) + M_\mu M_\nu\partial_\alpha(Q^2)\right].
\end{aligned}
\tag{5.49}
$$

We note the additional important calculations and definitions

$$
Z_{\mu\nu} = \sigma^\alpha_\mu\sigma^\beta_\nu Z_{\alpha\beta} + M_\mu\sigma^\alpha_\nu\partial_\alpha(N_\phi/s^2) - M_\mu\sigma^\alpha_\mu\partial_\alpha(N_\phi/s^2) \equiv {}^2Z_{\mu\nu} + M_\mu[\cdots] \tag{5.50}
$$

$$
W_{\mu\nu} = \gamma^\alpha_\mu\gamma^\beta_\nu W_{\alpha\beta} = \sigma^\alpha_\mu\sigma^\beta_\nu W_{\alpha\beta} + W_{\alpha\beta}M^\beta[\cdots] = \sigma^\alpha_\mu\sigma^\beta_\nu W_{\alpha\beta} \equiv {}^2W_{\mu\nu} \tag{5.51}
$$

which shows that W lives purely on the two manifold and Z lives on a three manifold defined by modding the full spacetime out by the axisymmetric Killing vector. We define $\mathcal{D}_\mu$ to be the covariant derivative with respect to the 2-metric σ and calculate the projected components of the Ricci tensor to be

$$
X^\mu X^{\nu 4}R_{\mu\nu} = -\frac{s}{Q}\mathcal{D}_\alpha(Q\mathcal{D}^\alpha s) + \frac{s^4}{4}{}^2Z_{\alpha\beta}{}^2Z^{\alpha\beta} - \frac{s^4}{2Q^2}\mathcal{D}_\alpha(N_\phi/s^2)\mathcal{D}^\alpha(N_\phi/s^2) \tag{5.52}
$$

$$
X^\mu(-Q^2M^\nu)^4R_{\mu\nu} = \frac{Q}{2s}\mathcal{D}_\alpha\left(\frac{s^3}{Q}\mathcal{D}^\alpha(N_\phi/s^2)\right) - \frac{s^2Q^2}{4}{}^2Z_{\alpha\beta}{}^2W^{\alpha\beta} \tag{5.53}
$$

$$
(-Q^2M^\mu)(-Q^2M^\nu)R_{\mu\nu} = \frac{Q}{s}\mathcal{D}_\alpha(s\mathcal{D}^\alpha Q) - \frac{s^2}{2}\mathcal{D}_\alpha(N_\phi/s^2)\mathcal{D}^\alpha(N_\phi/s^2) + \frac{Q^4}{4}{}^2W_{\alpha\beta}{}^2W^{\alpha\beta} \tag{5.54}
$$

$$
\sigma^\mu_\beta X^\nu R_{\mu\nu} = \frac{1}{2}\frac{s^3}{Q}\mathcal{D}_\alpha(Q/sZ^\alpha_\beta) \tag{5.55}
$$

$$
\sigma^\mu_\beta(-Q^2M^\nu)R_{\mu\nu} = -\frac{1}{2sQ}\mathcal{D}_\alpha(sQ^{32}W^\alpha_\beta) \tag{5.56}
$$

$$\sigma^{\mu}_{\beta}\sigma^{\nu}_{\delta}R_{\mu\nu} = {}^2R_{\beta\delta} - \mathcal{D}_{\beta}\mathcal{D}_{\delta}\ln(sQ) - \frac{s^2}{2}{}^2Z_{\delta\alpha}Z^{\alpha}_{\beta} + \frac{Q^2}{2}{}^2W_{\delta\alpha}{}^2W^{\alpha}_{\beta}$$
$$-\frac{1}{s^2}\mathcal{D}_{\beta}s\mathcal{D}_{\delta}s - \frac{1}{Q^2}\mathcal{D}_{\beta}Q\mathcal{D}_{\delta}Q - \frac{s^2}{2Q^2}\mathcal{D}_{\beta}(N_{\phi}/s^2)\mathcal{D}_{\delta}(N_{\phi}/s^2). \tag{5.57}$$

The scalar curvature is given via contraction of the previous by

$$^4R = {}^2R - \mathcal{D}_{\alpha}(\mathcal{D}^{\alpha}(sQ)/sQ) - \frac{1}{sQ}\mathcal{D}_{\alpha}(\mathcal{D}^{\alpha}(sQ)) - \frac{s^2}{4}{}^2Z_{\alpha\beta}{}^2Z^{\alpha\beta} - \frac{Q^2}{4}{}^2W_{\alpha\beta}{}^2W^{\alpha\beta}$$
$$-\frac{1}{s^2}\mathcal{D}_{\alpha}s\mathcal{D}^{\alpha}s - \frac{1}{Q^2}\mathcal{D}_{\alpha}Q\mathcal{D}^{\alpha}Q + \frac{s^2}{2Q^2}\mathcal{D}_{\alpha}(N_{\phi}/s^2)\mathcal{D}^{\alpha}(N_{\phi}/s^2).$$

These equations are relatively simple. The Einstein equations become a second order equation for s, an equation for N^{ϕ}, and an equation for q. They also produce two Maxwell like equations for $W_{\mu\nu}$ and $Z_{\mu\nu}$. The final Einstein equations give a two dimensional relativity on the 2-manifold defined with metricσ where $^2R_{\mu\nu}$ is the Ricci curvature defined with respect to $\sigma_{\mu\nu}$. This is similar to the results of Kaluza-Klein. Also one can show that circularity is recovered in the case that $Z_{\mu\nu} = W_{\mu\nu} = 0$. It is most natural to choose conformal gauge on the two-manifold. (i.e. $\sigma_{ij} = e^{\psi}\delta_{ij}$). In this gauge the two manifold scalar curvature and Ricci tensor take the form

$$R = -e^{-\psi}\Delta\psi \tag{5.58}$$

$$R_{ij} = \frac{1}{2}Rg_{ij} = -\frac{1}{2}\Delta\psi \tag{5.59}$$

where the Laplacian is with respect to the Euclidean metric (see for instance [15]). This makes the Einstein equations on the two manifold particularly simple.

5.4 GRMHD Stress Tensor

We have reduced the left hand side of the Einstein equations to equations given by two manifold and projection quantities. We need to preform a similar analysis for the matter part of Einstein equations.

Our stated interest is in magnetized neutron stars. Thus we consider the GRMHD stress tensor

$$\begin{aligned}\widetilde{T}_{\mu\nu} &= T_{\mu\nu} + \frac{b_\alpha b^\alpha}{4\pi} u_\mu u_\nu + \frac{b_\alpha b^\alpha}{8\pi}\gamma_{\mu\nu} - \frac{b_\mu b_\nu}{4\pi} \\ &= \left[\rho_0(1+\epsilon) + P + \frac{b_\alpha b^\alpha}{4\pi}\right] u_\mu u_\nu + \left[P + \frac{b_\alpha b^\alpha}{8\pi}\right]\gamma_{\mu\nu} - \frac{b_\mu b_\nu}{4\pi}. \end{aligned} \tag{5.60}$$

To begin let up make two convenient conditions. Let ξ_μ be any vector. Define

$$\hat{\xi} \equiv M^\mu \xi_\mu = \xi_t M^t + \xi_\phi M^\phi = \frac{1}{Q^2}[\xi_t - \xi_\phi(N_\phi/s^2)] \tag{5.61}$$

and

$$\widetilde{\xi} = Y^\mu \xi_\mu = \frac{1}{s^2}\xi_\phi. \tag{5.62}$$

We have the projections

$$\widetilde{T}_{\mu\nu}Y^\mu Y^\nu = \left[\rho_0(1+\epsilon) + P + \frac{b^2}{4\pi}\right](\widetilde{u})^2 + \left[P + \frac{b^2}{8\pi}\right]\frac{1}{s^2} - \frac{1}{4\pi}(\widetilde{b})^2 \tag{5.63}$$

$$\widetilde{T}_{\mu\nu}M^\mu M^\nu = \left[\rho_0(1+\epsilon) + P + \frac{b^2}{4\pi}\right](\hat{u})^2 + \left[P + \frac{b^2}{8\pi}\right]\frac{1}{Q^2} - \frac{1}{4\pi}(\hat{b})^2 \tag{5.64}$$

$$\widetilde{T}_{\mu\nu}M^\mu Y^\nu = \left[\rho_0(1+\epsilon) + P + \frac{b^2}{4\pi}\right]\widetilde{u}\hat{u} - \frac{1}{4\pi}\widetilde{b}\hat{b} \tag{5.65}$$

and similar projections for two-manifold quantities. These calculations give part of the righthand side of Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$. We now turn attention to the matter equations

$$0 = \nabla_a \widetilde{T}^{ab} = \nabla_a T^{ab} + \frac{1}{4\pi}[b^2 \nabla_a (u^a u^b) + u^a u^b \partial_a b^2] + \frac{1}{8\pi} g^{ab} \partial_a b^2 - \frac{1}{4\pi} \nabla_a (b^a b^b)$$

$$= \left[\nabla_a T^{ab} + \frac{b^2}{4\pi}(u^a \nabla_a u^b) + \frac{g^{ab}}{8\pi} \partial_a b^2 \right] - \frac{1}{4\pi} \nabla_a (b^a b^b). \tag{5.66}$$

Lowering the free index we have

$$0 = \left[\nabla_a T^a_b + \frac{b^2}{4\pi}(u^a \nabla_a u_b) + \frac{1}{8\pi} \partial_b b^2 \right] - \frac{1}{4\pi} \nabla_a (b^a b_b) \tag{5.67}$$

which we may write in differential form as

$$0 = \left[\rho_0 + \rho_i + P + \frac{b^2}{4\pi} \right] \left[d \ln u^t - u^t u_\phi d\Omega \right] + d \left[P + \frac{b^2}{8\pi} \right] - \frac{1}{4\pi} \nabla_a (b^a \mathbf{b}). \tag{5.68}$$

Finally, we explicitly calculate the final term

$$\nabla_a (b^a b_b) = b^a \partial_a b_b + b_b \partial_a b^a - b^2 \alpha_{,b} - \frac{1}{r} \delta^\theta_b b_\theta b^\theta$$
$$+ b_b \left(b^r [\gamma_{,r} + \frac{2}{r} + 2\alpha_{,r}] + b^\theta [\gamma_{,\theta} + \cot\theta + 2\alpha_{,\theta}] \right). \tag{5.69}$$

Additional work needs to be done to write the last term in differential form. We now turn attention to our set of equations which govern the evolution of the magnetic field.

5.5 Maxwell Equations

The Maxwell equations will govern the magnetic field b^a. We take F^{ab} as the Maxwell stress tensor and project the Maxwell equations $J^b = \nabla_a F^{ab}$ in a manner similar to our previous projections of the Einstein equations. We compute

$$J^b = \nabla_a F^{ab} = \partial_a F^{ab} + \Gamma^a_{ac} F^{cb} = \frac{1}{\sqrt{-\gamma}} \partial_a (\sqrt{-\gamma} F^{ab}) \tag{5.70}$$

We now compute the projection

$$\sqrt{-\gamma} J^b = \gamma^c_a \partial_c (\sqrt{-\gamma} \gamma^a_d \gamma^b_c F^{de})$$

$$= \gamma^c_a \partial_c \Big(\sqrt{-\gamma}\ {}^2F^{ab} + \Big[Y^b s^2 Y_e \left\{ \sigma^a_d - Q^2 M^a M_d \right\} - M^b Q^2 M_e \left\{ \sigma^a_d - s^2 Y^a Y_d \right\}$$
$$+ \sigma^b_e \left\{ s^2 Y^a Y_d - Q^2 M^a M_d \right\} \Big] \sqrt{-\gamma} F^{de} \Big)$$

$$= \partial_a (\sqrt{-\gamma}^2 F^{ab}) + s^2 Y^b \partial_a \left[\sqrt{-\gamma} F^{de} Y_e \sigma^a_d \right] - Q^2 M^b \partial_a \left[\sqrt{-\gamma} F^{de} M_e \sigma^a_d \right] \tag{5.71}$$

where we have used the useful identity

$$\partial_a \ln \sqrt{-\gamma} = \partial_a \ln (sQ\sqrt{\sigma}). \tag{5.72}$$

We recall that $F_{ab} = \partial_a A_b - \partial_b A_a$ and note $F^{ab} = \gamma^{ac}\gamma^{bd}[\partial_c A_d - \partial_d A_c]$. We project the Maxwell equations to find

$$\sqrt{-\gamma} Y_b J^b = \partial[\sqrt{-\gamma} F^{de} Y_e \sigma^a_d] + Y_b \partial_a (\sqrt{-\gamma}\ {}^2F^{ab}) \tag{5.73}$$

$$= \frac{\sqrt{-\gamma}}{s^2} \sigma^{ah} \partial_a \partial_h A_\phi + \partial_h (A_\phi) \partial_a \left[\frac{\sqrt{-\gamma}}{s^2} \sigma^{ah} \right] + Y_b \partial_a \left(\sqrt{-\gamma}\ {}^2F^{ab} \right) \tag{5.74}$$

where we have defined ${}^2F^{ab} = \sigma^a_c \sigma^b_d F^{cd}$. If we choose σ to be a conformally Euclidean metric in polar coordinates, it takes the form

$$\sigma_{ab} = e^\psi \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{5.75}$$

and the previous equation becomes

$$\partial_r^2 A_\phi + \frac{1}{r^2}\partial_\theta^2 A_\phi = e^{2\psi} J_\phi - \sigma^{ab}(\partial_a A_\phi)\partial_b \ln\left[\frac{\sqrt{-\gamma}}{s^2}e^{-2\psi}\right] + Y_b\partial_a\left(\sqrt{-\gamma}^{\,2}F^{ab}\right) \quad (5.76)$$

which determines A_ϕ. A parallel calculation shows that projecting along the axisymmetric Killing vector gives

$$\sqrt{-\gamma}M_b J^b = \partial_a[\sqrt{-\gamma}F^{de}Y_e\sigma^a_d] + M_b\partial_a[\sqrt{-\gamma}^{\,2}F^{ab}]$$

$$= \frac{\sqrt{-\gamma}}{Q^2}\sigma^{ah}\left[\partial_a\partial_h A_t - \partial_a\left(\frac{N_\phi}{s^2}\partial_h A_\phi\right)\right]$$

$$+ \left[\partial_h A_t - \frac{N_\phi}{s^2}\partial_h A_\phi\right]\partial_a\left[\frac{\sqrt{-\gamma}}{Q^2}\sigma^{ah}\right] + M_b\partial_a\left[\sqrt{-\gamma}^{\,2}F^{ab}\right] \quad (5.77)$$

This is an equation involving both A_t and A_ϕ. If we use the previous equation to eliminate A_ϕ as well as our previous coordinates, we find

$$\partial_r^2 A_t + \frac{1}{r^2}\partial_\theta^2 A_t = \frac{N_\phi}{s^2}e^{2\psi}J_\phi - \frac{N_\phi}{s^2}\partial_r A_\phi \partial_r \ln\left[\frac{\sqrt{-\gamma}}{s^2}e^{-2\psi}\right] - \frac{N_\phi}{s^2}\frac{\partial_\theta A_\phi}{r^2}\partial_\theta \ln\left[\frac{\sqrt{-\gamma}}{s^2}e^{-2\psi}\right] \quad (5.78)$$

$$+ \frac{N_\phi}{s^2}Y_b\partial_a(\sqrt{-\gamma}^{\,2}F^{ab}) + e^{2\psi}\partial_h A_\phi \partial_a(N_\phi/s^2)\sigma^{ah} + e^{2\psi}Q^2 M_b J^b \quad (5.79)$$

$$- \frac{e^{2\psi}Q^2}{\sqrt{-\gamma}}\left[\partial_h A_t - \frac{N_\phi}{s^2}\partial_h A_\phi\right]\partial_a\left[\sqrt{-\gamma}Q^2\sigma^{ab}\right] - \frac{e^{2\psi}Q^2 M_b}{\sqrt{-\gamma}}\partial_a\left[\sqrt{-\gamma}^{\,2}F^{ab}\right]. \quad (5.80)$$

Finally, we project the Maxwell equations into the two manifold to find

$$\sqrt{-\gamma}\sigma_{bc}J^b = \sigma_{bc}\partial_a(\sqrt{-\gamma}^{\,2}F^{ab})$$

$$= \sqrt{-\gamma}\sigma_{bc}\sigma^{ae}\sigma^{bf}\partial_a\left[\partial_e A_f - \partial_f A_e\right] + \sigma_{bc}(\partial_e A_f - \partial_f A_e)\partial_a(\sqrt{-\gamma}\sigma^{ae}\sigma^{bf}) \tag{5.81}$$

$$\equiv \sqrt{-\gamma}\sigma^{ae}\partial_a\left[\partial_e A_c - \partial_c A_e\right] + O_c. \tag{5.82}$$

where there are no second derivatives of A_i in O_c. Again choosing σ to be a conformally Euclidean metric in polar coordinates and letting c range over r and θ, we find

$$\partial_\theta^2 A_r - \partial_r\partial_\theta A_\theta = r^2\sigma_{br}J^b - \frac{O_r}{\sqrt{-\gamma}} \tag{5.83}$$

$$\partial_\theta^2 A_\theta - \partial_r\partial_\theta A_r = r^2\sigma_{b\theta}J^b - \frac{O_\theta}{\sqrt{-\gamma}}. \tag{5.84}$$

We can use the Maxwell gauge constraint

$$0 = \nabla_\mu A^\mu = \mathcal{D}_\mu A^\mu + A^\mu\partial_\mu \ln(sQ) \tag{5.85}$$

to write the A_r and A_θ equations as linear equations. This now completes the analysis of deriving the equations for full axisymmetric GRMHD.

5.6 Conclusions

We first considered the analysis of [5] in which differentially rotating unmagnetized neutron stars were considered. In this case the Einstein equations admitted a form that could be solved by an iterated Green's function method. We proceeded to extend this analysis and that of [3] to the case of differentially rotating non-circular magnetized neutron stars. This was significantly more complicated since we have a general four metric. However, decomposing the metric along our spacelike and timelike Killing vectors led to a relatively aesthetic form of the Einstein equations with respect to the two metric. We are now in a position to apply numerical integration of these equations for later work.

Chapter 6

Conclusions

This thesis has predominately been concerned with investigations related to evolutionary problems in black hole physics. After a brief summary of the Einstein equations, we proceeded to consider the linear stability of black strings and p-branes (a charged analogue of black strings). It was concluded that black strings always admit linearly unstable modes; however, certain classes of p-branes dependent on a coupling constant between a scalar dilaton and an n-form field can be stabilized for a sufficiently large amount of charge.

We then considered the eigenvalue problem for the GRMHD equations and computed all spectral information. Moreover, we established degeneracy conditions on the eigenvectors and provided scaling factors that should eliminate such problems. Finally, we derived the Einstein and matter equations for a general stationary axisymmetric spacetime with the GRMHD stress tensor without assuming the circularity condition.

Further work needs to be done on each of these projects. First, it would be interesting to understand why the threshold masses for unstable black p-branes increase for large values of μ for some fixed a. This seems to imply that for some a, applying a large amount of charge to a certain p-brane makes it more unstable than lesser amounts of charge. Can this be explained? Also, can p-brane solutions to more general low energy string theories be considered? Secondly, our GRMHD spectral data needs to be incorporated into current numerical schemes to fix spurious boundary waves. Finally, our equations for the general axisymmetric initial value problem need to be numerically solved.

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Appendix A

Numerical Code

This code is written for Mathematica 6.0, and is used to solve the perturbation equations mentioned in chapter two.

```
   (* The following is a numerical method for integrating perturbation
equations of a class of place p-branes as referenced in Chapter 2. It
searches for a value m such that P<0 by integrating the perturbation
equations with a fixed value of mu.  When m has been found, mu is
incremented and the process repeats.  The output is given in  sols. *)
   j = 1;                               (* Initialize secondary loop index j *)
   stor = ConstantArray[0, {2000, 3}]; (* An array for main loop data *)
   sols = ConstantArray[0, {600, 3}];  (* An array for secondary loop data *)
   While[j < 500,           (* Begin Secondary loop *)
   i = 1;                   (* Initialize main loop index i *)
   P = 1;                   (* Initialize the value of P *)
   \[Mu] = j/100 - .01;     (* Increment mu by hundreths *)
   While[P > 0,             (* Begin main loop *)
   m = i/1000 - .001;       (* Increment m by thousandths *)
   a = 1;
   d = 3;
   dt = 5;                                         (* Define d tilde *)
   k = 2^5/(dt + 1 + 4 dt/tri Sinh[\[Mu]]^2); (* Define k for ADM mass 2^5 *)
   rh = k^(1/dt);                                  (*Define event horizon value *)
   tri = a^2 + 2 d dt/8;
   \[Epsilon] = 10^(-6); (* Define horizon offstep *)
   rm = 500;             (* Define endpoint of numerical integration *)

    (* Input perturbation equations *)
```

```
eqn1 = r (r^dt - k) \[Phi]''[r] + ((dt + 1) r^dt - k) \[Phi]'[r] -
m^2 r^(dt + 1 - 4 dt/tri) (r^dt + k Sinh[\[Mu]]^2)^(4/
tri) \[Phi][r] - dt k \[Psi]'[r] == 0;

W = dt (r^dt - k) (r^dt + k Sinh[\[Mu]]^2) -
2/tri (a^2 - 2 dt^2/8) Sinh[\[Mu]]^2 (r^dt - k)^2 +
dt k Cosh[\[Mu]]^2 r^dt;

eqn2 =
r^2 (r^dt - k)^2 (r^dt + k Sinh[\[Mu]]^2) \[Psi]''[r] +
r (r^dt - k)^2 (2 dt (r^dt - 2/tri k Sinh[\[Mu]]^2) - (dt - 3) (r^dt +
k Sinh[\[Mu]]^2)) \[Psi]'[r] - (m^2 r^(dt + 2 - 4 dt/tri) (r^dt -k)
(r^dt + k Sinh[\[Mu]]^2)^(1 + 4/tri) + dt k (W +
2/tri (2 dt^2 + (dt + 3) (a^2 - 2 dt^2/8)) Sinh[\[Mu]]^2 (r^dt - k)^2))
\[Psi][r] +dt k W \[Phi][r] == 0;

 (* Input boundary conditions II *)
phibnd = 1 + m^2 rh^2/dt Cosh[\[Mu]]^(8/tri) \[Epsilon]/rh;
phiderbnd = m^2 Cosh[\[Mu]]^(8/tri) rh/dt;

 (* Solve equations for boundary conditions I *)
q = NDSolve[{eqn1,eqn2, \[Phi][rh + \[Epsilon]] == \[Epsilon]/2,
\[Phi]'[rh + \[Epsilon]] == 1/2, \[Psi][rh + \[Epsilon]] == \[Epsilon]/2,
\[Psi]'[rh + \[Epsilon]] == 1/2}, {\[Phi], \[Psi]},
{r,rh + \[Epsilon], 500}, StartingStepSize -> 10^(-6)];

 (* Solve equations for boundary conditions II *)
s = NDSolve[{eqn1, eqn2, \[Phi][rh + \[Epsilon]] ==
phibnd, \[Phi]'[rh + \[Epsilon]] == phiderbnd, \[Psi][rh + \[Epsilon]] ==
1, \[Psi]'[rh + \[Epsilon]] == 0}, {\[Phi], \[Psi]},
 {r,rh + \[Epsilon], 500}, StartingStepSize -> 10^(-6)];

 (* Evaluate numerical results near infinity *)
ph1 = Evaluate[{\[Phi][rm], \[Psi][rm]} /. q][[1, 1]];
ps1 = Evaluate[{\[Phi][rm], \[Psi][rm]} /. q][[1, 2]];
ph2 = Evaluate[{\[Phi][rm], \[Psi][rm]} /. s][[1, 1]];
ps2 = Evaluate[{\[Phi][rm], \[Psi][rm]} /. s][[1, 2]];

P = ph1 ps2 - ph2 ps1;  (* Compute P *)

 (* Store Data *)
stor[[i, 1]] = m;
stor[[i, 2]] = P;
stor[[i, 3]] = \[Mu];

i++;    (* End loop if P<0 increment mu and proceed again *)
   ]
```

```
 sols[[j, 1]] = N[stor[[i - 1, 1]]];
 sols[[j, 2]] = N[stor[[i - 1, 2]]];
 sols[[j, 3]] = N[stor[[i - 1, 3]]];

 (* Print numerical progress *)
Print[``-------------------------'']; 
Print[ ``m    ----    mu'']; 
Print[``-------------------------'']; 
Print[Row[{N[sols[[j, 1]]], N[sols[[j, 3]]]}, ``    ----    '']];

 j++; ]
```